Long Cycles in 2-Connected Triangle-Free Graphs

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Abstract

Dirac showed that a 2–connected graph of order n with minimum degree δ has circumference at least $\min\{2\delta, n\}$. We prove that a 2–connected, triangle-free graph G of order n with minimum degree δ either has circumference at least $\min\{4\delta-4, n\}$, or every longest cycle in G is dominating. This result is best possible in the sense that there exist bipartite graphs with minimum degree δ whose longest cycles have length $4\delta - 4$, and are not dominating.

1 Introduction

In this note we study long cycles in 2-connected triangle-free graphs. Let C be a longest cycle in such a graph G. We investigate the length of C as a function of the minimum vertex degree of G, as well as the structure of $G - C$.

We begin with some standard definitions. The specific terminology and notation required for the proof of our main result is given in the next section. A good reference for any undefined terms is [18].

We consider only finite undirected graphs without loops or multiple edges. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. Then G is *hamiltonian* if it has a *Hamilton cycle*, i.e., a cycle containing all of its vertices. It has a *Hamilton path* if it has a path containing all of its vertices. A graph G is said to be *hamiltonian connected* if every pair of vertices in G are the endvertices of a Hamilton path. We use $\kappa(G)$ for the vertex connectivity of G, $\delta(G)$ for the minimum vertex degree of G

and $\alpha(G)$ to denote the cardinality of a largest set of independent vertices in G. Let $\omega(G)$ represent the number of components of G. We say G is *t-tough* if $|S| \geq t \cdot \omega(G-S)$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. A cycle C in G is called a *dominating cycle* if every edge of G has at least one of its endvertices on C. The circumference of G, denoted $c(G)$, is the number of vertices on a longest cycle in G. If no ambiguities are likely to arise, we frequently omit any explicit reference to the graph G by simply writing δ, κ, etc. We also sometimes identify a subgraph with its vertex set, e.g., use C for $V(C)$, etc. If $H \subseteq V(G)$ we denote the subgraph induced by H as $\langle H \rangle$.

First, a little background. We begin with a well-known theorem of Dirac [4].

Theorem 1.1. *Let G be a 2-connected graph on n vertices. Then $c(G) \geq min\ \{n, 2\delta\}$.*

This result was recently strengthened in [12].

Theorem 1.2. *Let G be a 2-connected graph on n vertices. Then either $c(G) \geq min\{n,\ n+\delta-\alpha,\ 3\delta-3\}$ or $G \in \mathcal{F}^*$.*

The class of graphs $\mathcal{F}^*$ referred to above is a subclass of the well-known collection of graphs described by Watkins and Mesner in [17]. All of these graphs have connectivity two and are recognizable in polynomial time.

In [2], Bondy studied the relationship between a longest cycle C in a k-connected graph G and the structure of the components of $G-C$. His paper, as well as Theorem 1.1, have led many authors to follow this line of research [4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16].

Our main result is closely related to the following conjecture of Jung [11].

Conjecture 1.3. *Let C be a longest cycle in a k-connected graph, $k \geq 2$. If $G-C$ contains a path with at least $k-1$ vertices, then $|C| \geq k(\delta-k+2)$.*

Note that Conjecture 1.3 reduces to Theorem 1.1 when $k=2$. While partial solutions have been found for $k \leq 6$, Conjecture 1.3 remains open.

In this paper, we establish a result analogous to Conjecture 1.3 for 2-connected triangle-free graphs. Note that in Conjecture 1.3 it is not necessary to assume $\delta(G)$ is large. The same is true of our result, given in Theorem 1.7. By contrast, a number of results on long cycles in triangle-free graphs have required $\delta(G)$ to be large. For example, Brandt [3] was able to determine all cycle lengths in non-bipartite triangle-free graphs on n vertices with $\delta > n/3$. Define $C(G)$ to be the set of cycle lengths of a graph G.

Theorem 1.4. [3] *Let $G \neq C_5$ be a triangle-free, nonbipartite graph of order n. If $\delta > n/3$, then $C(G) = \{4, 5, ..., r\}$, where $r = min\ \{n, 2(n-\alpha)\}$.*

As a corollary, Brandt obtained the following.

Corollary 1.5. [3] *Let $G \neq C_5$ be a triangle-free non-bipartite graph of order n. If $\delta > 3n/8$, then G contains a cycle of length l, for $4 \leq l \leq n$.*

A few years earlier, Aung [1] considered the structure of $G - C$, where C is a longest cycle in a 2-connected triangle-free graph G. We state his main result below. Note that it requires a minimum degree condition.

Theorem 1.6. *Let C be a longest cycle in a 2-connected triangle-free graph G on n vertices. If $\delta \geq (n+6)/6$, then $G - C$ cannot have two edges.*

Our main result is the following.

Theorem 1.7. *Let G be a 2–connected triangle–free graph of order n. Then $c(G) \geq min\{n, 4\delta - 4\}$ or every longest cycle in G is a dominating cycle.*

The proof of Theorem 1.7 is given in Section 3.

2 Preliminaries

The proof of Theorem 1.7 requires some notation and terminology. Let C be a cycle in G. We denote by $\overrightarrow{C}$ the cycle C with a given orientation. Suppose $u, v \in V(C)$. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. If $U \subseteq V(C)$ then $U^+ = \{u^+ \mid u \in U\}$, and U^- is defined similarly. Further define $u^{+2} = (u^+)^+$ and $u^{-2} = (u^-)^-$. The set of all vertices strictly between u and v on $\overrightarrow{C}$ is symbolized by $(u\overrightarrow{C}v)$. If the vertices u and v are to be included in this set, it will be written as $[u\overrightarrow{C}v]$. Similarly, (uPv) and $[uPv]$ denote sets of vertices along a path P.

The neighborhood of a vertex h in A, denoted $N_A(h)$, is the set of all vertices in A that are adjacent to h. We define $d_A(h) = |N_A(h)|$ and $N_A^+(h) = \{c \in V(A) \mid hc^- \in E(G)\}$, and let $N_A(B)$ denote the set of all vertices in A adjacent to some vertex in B.

Let H be a component of $G - C$. Following Jung [8], we now define the relative connectivity of H in G. Consider all nonempty subsets K of $V(H)$ having $N_G(K) \cup K \neq V(G)$. We define $\kappa_H = \min |N_{G-K}(K)|$ over all such K. Note that $\kappa_H \geq \kappa(G)$.

If $X = \{x \in V(C) \mid \ d_H(x) \geq 2\}$ and $Y = \{y \in H \mid N_C(y) \neq \emptyset \text{ and } N_C(y) \cap X = \emptyset\}$, then let $\mu_H = |X \cup Y|$. Note that $\mu_H \geq \min\{\kappa_H, |H|\}$.

Now let H be a subgraph (not necessarily a cycle) of G. If for every pair of vertices $x, y \in N_{G-H}(H)$ for which $|N_H(\{x, y\})| \geq 2$ there exists a hamiltonian path $P = P[x', y']$ of H such that $x \in N(x')$ and $y \in N(y')$, then H is said to be *strongly linked* in G. Otherwise H is said to be *weakly linked* in G.

A 3–matching M from H to C is a matching that consists of three distinct edges of G, $\{x_1y_1, x_2y_2, x_3y_3\}$, where $x_1, x_2, x_3 \in V(H)$ and $y_1, y_2, y_3 \in V(C)$.

A subgraph S of G is called a *stronghold* in G if S is complete and the $N_{G-S}(u) = N_{G-S}(v)$ for all $u, v \in V(S)$.

Before proving our main theorem we present several useful results.

Lemma 2.1. [1]
Let G be a triangle-free graph on n vertices and let P be a path in G with end-vertex u. Then $|V(P)| \geq 2d_P(u)$.

Theorem 2.2. [1]
Let a and b be distinct vertices and P a longest $ab-$path in the 2-connected, triangle-free graph G. Then each component H of $G - P$ has a vertex w such that $|V(P)| \geq 2d(w) - 1 \geq 2\delta - 1$.

Theorems 2.3 to 2.7 below are due to Jung and appear in [8]. In each theorem C is a longest cycle in G and H is a component of $G - C$.

Theorem 2.3. [8]
Let $\kappa_H \geq 3$ and suppose H contains a cut-vertex. Then there exist nonadjacent vertices v, w in H such that $|V(C)| \geq 2d(v) + 2d(w) - 4 + \epsilon$, for some nonnegative integer ϵ.

Theorem 2.4. [8]
Let H be 2-connected and contain no hamiltonian cycle. If $\kappa_H \geq 3$ then there exist nonadjacent vertices v, w in H such that $|V(C)| \geq 2d(v)+2d(w)$.

Theorem 2.5. [8]
Let H be hamiltonian with $\kappa_H \geq 3$. If H is weakly linked in G, then there exist nonadjacent vertices v, w in H such that $|V(C)| \geq 2d(v) + 2d(w) + min\{|V(H)|/2, 6\}$.

Theorem 2.6. [8]
Let H be strongly linked in G and suppose $\kappa_H \geq 2$. Then there exists a vertex v in H such that $|V(C)| \geq k(d(v)-k+3)+(\kappa_H-k)(|V(H)|+1-k)$ whenever $0 \leq k \leq |V(H)|+1$, unless H is a stronghold in G.

Theorem 2.7. [8]
Let H contain a hamiltonian cycle and let $\kappa_H \geq 2$. If H is strongly linked in G but not hamiltonian connected, then there exist nonadjacent vertices v, w in H such that $|V(C)| \geq 2d(v) + 2d(w) + (\mu_H - 2)(|V(H)| - 3) - 6$.

3 Proof of Theorem 1.7

Let G be a 2–connected, triangle–free graph on n vertices and suppose that C is a longest cycle in G. If $\delta = 2$, then the result is immediate from Theorem 1.1. Hence we assume $\delta \geq 3$.

If all components of $G - C$ are of order 1, then C is dominating and we are done. So we begin by establishing the theorem when $G - C$ contains a component of order 2 or 3.

Claim: If there exists a component H of $G-C$ with $2 \leq |V(H)| \leq 3$, then $c(G) \geq 4\delta - 4$.

Proof of Claim: Suppose H is a component of $G-C$ with $V(H) = \{x, y\}$. Let $N_C(x) = U = \{u_1, u_2, \ldots, u_s\}$ and $N_C(y) = W = \{w_1, w_2, \ldots, w_t\}$, where $s,\ t \geq \delta - 1 \geq 2$.

Since G is triangle–free and C is a longest cycle in G, the sets U, U^+, W, W^+ are pairwise disjoint. Thus

$$\begin{aligned} |V(C)| &\geq |U \cup U^+ \cup W \cup W^+| \\ &\geq 4(\delta - 1) \\ &= 4\delta - 4. \end{aligned}$$

Now suppose H is a component of $G-C$ with $V(H) = \{x, y, z\}$ and $E(H) = \{xy, yz\}$. Let $N_C(x) \cup N_C(z) = U = \{u_1, u_2, \ldots, u_s\}$ and let $N_C(y) = W = \{w_1, w_2, \ldots, w_t\}$, where $s \geq \delta - 1 \geq 2$ and $t \geq \delta - 2 \geq 1$. Again we have that the sets U, U^+, W, W^+ are pairwise disjoint. Since $U \cap W = \emptyset$, there exist $i \in \{1, 2, \ldots, s\}$ and $j \in \{1, 2, \ldots, t\}$ such that $w_j \in (u_i \overrightarrow{C} u_{i+1})$, where the indices of vertices in U (resp., W) are modulo s (resp., t). We have at once that $u_{i+1}^- \notin U \cup U^+ \cup W \cup W^+$. If we take w_j to be the vertex in $W \cap (u_i \overrightarrow{C} u_{i+1})$ closest to u_i along $\overrightarrow{C}$, then we also have $w_j^- \notin U \cup U^+ \cup W \cup W^+$. Thus

$$\begin{aligned} |V(C)| &\geq |U \cup U^+ \cup W \cup W^+ \cup \{u_{i+1}^-, w_j^-\}| \\ &\geq 2d_C(x) + 2d_C(y) + 2 \\ &\geq 2(\delta - 1) + 2(\delta - 2) + 2 \\ &= 4\delta - 4. \end{aligned}$$

which completes the proof of the claim. $\square$

Now let H be a component of $G-C$ with $|V(H)| \geq 4$. The remainder of the proof is divided into two cases.

Case 1: $\kappa(G) \geq 3$.

Case 1a: H is hamiltonian connected.

Suppose there exists a vertex $v \in H$ having $d_H(v) \geq \delta - 1$. Since $\kappa(G) \geq 3$ there exists a 3-matching $M = \{p'p, q'q, r'r\}$ from H to C, where $p', q', r' \in V(H)$ and $p, q, r \in V(C)$. Let P be the Hamilton path in H from p' to q'. Since $v \in V(P)$ and G is triangle-free, we deduce from Lemma 2.1 that $|[p'Pv]| \geq 2d_1$ and $|[vPq']| \geq 2d_2$, where d_1 and d_2 represent the degree of v on $[vPp']$ and $[vPq']$ respectively. Thus $|V(P)| = |[p'Pv]| + |[vPq']| - 1 \geq 2d_H(v) - 1 \geq 2(\delta - 1) - 1 = 2\delta - 3$. The choice of

$\overrightarrow{C}$ implies $|(p\overrightarrow{C}q)| \geq 2\delta - 3$. Analogous arguments yield $|(q\overrightarrow{C}r)| \geq 2\delta - 3$ and $|(r\overrightarrow{C}p)| \geq 2\delta - 3$. Therefore

$$\begin{aligned} |V(C)| &\geq |(p\overrightarrow{C}q)| + |(q\overrightarrow{C}r)| + |(r\overrightarrow{C}p)| + |\{p,q,r\}| \\ &\geq 3(2\delta - 3) + 3 \\ &= 6\delta - 6 \\ &\geq 4\delta - 4. \end{aligned}$$

Now suppose $d_C(v) \geq 2$ for all $v \in H$, and let $xy \in E(H)$. Let $N_C(x) = \{x_1, x_2, \ldots, x_m\}$ and $N_C(y) = \{y_1, y_2, \ldots, y_l\}$, indices modulo m and l respectively. Without loss of generality, assume $m \geq l \geq 2$. Since G is triangle-free and C is a longest cycle in G, the sets $N_C(x), N_C^+(x), N_C(y)$, and $N_C^+(y)$ are pairwise disjoint.

Since $N_C(x) \cap N_C(y) = \emptyset$, there exists $h \in \{1, 2, \ldots, m\}$ such that $(x_h\overrightarrow{C}x_{h+1})$ contains one or more neighbors of y. Let y_j (resp. y_k) $\in (x_h\overrightarrow{C}x_{h+1})$ be the neighbor of y closest to x_h (resp. x_{h+1}) along $\overrightarrow{C}$, where we allow the possibility that $y_j = y_k$.

Let P denote a Hamilton path in H joining x and y. Since $d_H(y) \geq \delta - l$, we have $|V(P)| \geq 2(\delta - l)$, by Lemma 2.1. This implies, given the choice of $\overrightarrow{C}$, that $|(x_h\overrightarrow{C}y_j)|, |(y_k\overrightarrow{C}x_{h+1})| \geq 2(\delta - l)$. Therefore

$$\begin{aligned} |V(\overrightarrow{C})| &\geq |(x_h\overrightarrow{C}y_j) \cup (y_k\overrightarrow{C}x_{h+1})| + |N_C(x) \cup (N_C^+(x) - \{x_h^+\})| \\ &\quad + |N_C(y) \cup (N_C^+(y) - \{y_k^+\})| \\ &\geq 2(2(\delta - l)) + (2m - 1) + (2l - 1) \\ &= 4\delta - 2l + 2m - 2 \\ &\geq 4\delta - 2. \qquad \square \end{aligned}$$

Case 1b: H is *not* hamiltonian connected.

We first apply Theorems 2.3 - 2.7 to show that unless H has a very specific structure, $c(G) \geq 4\delta - 4$. Note that since G is 3-connected, $\kappa_H \geq 3$. If H has a cut-vertex, then $c(G) \geq 4\delta - 4$ by Theorem 2.3. Hence we may assume H is 2-connected. If H does not contain a Hamilton cycle, then Theorem 2.4 implies that $c(G) \geq 4\delta$. Otherwise, H is hamiltonian and we need to consider how H is linked to $\overrightarrow{C}$.

If H is weakly linked to C, then our conclusion is immediate from Theorem 2.5. So suppose H is strongly linked to C and $\kappa_H \geq 4$. Since H is not hamiltonian connected by assumption, and therefore not a stronghold, Theorem 2.6 with $k = 4$ yields

$$|V(C)| \geq 4(\delta - 1) + (\kappa_H - 4)(|V(H)| - 3) \geq 4\delta - 4.$$

Thus we assume H is strongly linked to C and $\kappa_H = 3$. If $|V(H)| > 4$, then since $\mu_H \geq \kappa_H$, $c(G) \geq 4\delta - 4$ by Theorem 2.7. Thus we may assume that H is an induced cycle on four vertices.

Let $V(H) = \{x, y, z, q\}$ and $E(H) = \{xy, yz, zq, qx\}$. Define $U = N_C(x) \cup N_C(z) = \{u_1, u_2, \ldots, u_m\}$ and $W = N_C(y) \cup N_C(q)$. Note that $U \cap W = U^+ \cap W = U \cap W^+ = \emptyset$.

Since $\delta > 2$ we are guaranteed that U and W are non-empty. If $|U| = \{u\}$ and $W = \{w\}$, then G is 2-connected, a contradiction. Hence we may assume there exists $i \in \{1, \ldots, m\}$ and a vertex $w \in W$ such that $w \in (u_i \overrightarrow{C} u_{i+1})$, indices modulo m. We may also assume that w is the vertex in $W \cap (u_i \overrightarrow{C} u_{i+1})$ closest to u_i along $\overrightarrow{C}$. The choice of $\overrightarrow{C}$ implies that $\{w^-,\ w^{-2},\ w^{-3},\ u_{i+1}^-\} \notin U \cup U^+ \cup W \cup W^+$.

Therefore

$$\begin{aligned} |V(C)| &\geq |U \cup U^+ \cup W \cup W^+| + |\{w^-,\ w^{-2},\ w^{-3},\ u_{i+1}^-\}| \\ &\geq 4(\delta - 2) + 4 \\ &\geq 4\delta - 4. \qquad \square \end{aligned}$$

Hence the theorem is valid for $\kappa(G) \geq 3$.

Case 2: $\kappa(G) = 2$.

Let $\{u, v\}$ be a 2-cut in G. Let T_1 and T_2 be two components of $G - \{u, v\}$, and let P_i be a longest path in $\langle T_i \cup \{u, v\} \rangle$, for $i = 1, 2$. We will show $|V(P_1)| \geq 2\delta - 1$. An identical argument shows that $|V(P_2)| \geq 2\delta - 1$, and thus G will contain a cycle C through u and v with $|C| \geq 2(2\delta - 1) - 2 = 4\delta - 4$.

Let $T \doteq \langle T_1 \cup \{u, v\} \rangle$. Since G is 2-connected, any cut-vertex of T must occur in $(u \overrightarrow{P_1} v)$. If T has cut-vertices, let $s_1, \ldots, s_r$ denote the cut-vertices of T in order on $(u \overrightarrow{P_1} v)$, and take $s_0 = u$ and $s_{r+1} = v$. If T is 2-connected, take $s_0 = u$ and $s_1 = v$.

Note that $\{s_j, s_{j+1}\}$ is a 2-cut for G, for some $j \in \{0, \ldots, r\}$. If T is 2-connected, this is trivial since $\{s_0, s_1\} = \{u, v\}$. If T has cut-vertices, and $(s_j \overrightarrow{P_1} s_{j+1}) \neq \emptyset$ for some $j \in \{0, \ldots, r\}$, then $\{s_j, s_{j+1}\}$ is a 2-cut in G. But for any $j \in \{1, \ldots, r\}$, either $(s_{j-1} \overrightarrow{P_1} s_j) \neq \emptyset$ or $(s_j \overrightarrow{P_1} s_{j+1}) \neq \emptyset$, since otherwise $d_G(s_j) = 2$, contradicting $\delta \geq 3$.

If there exist components $H_1, \ldots, H_l$ of $G - [s_j \overrightarrow{P_1} s_{j+1}]$ not containing u or v, then $B \doteq \langle [s_j \overrightarrow{P_1} s_{j+1}] \cup H_1 \cup \cdots \cup H_l \rangle$ is a 2-connected block with longest $s_j s_{j+1}$-path $[s_j \overrightarrow{P_1} s_{j+1}]$. Applying Theorem 2.2 to B, we conclude that for any $i \in \{1, \ldots, l\}$, there exists $w \in H_i$ with $|V(P_1)| \geq |[s_j \overrightarrow{P_1} s_{j+1}]| \geq 2d(w) - 1 \geq 2\delta - 1$. If there are no such components of $G - [s_j \overrightarrow{P_1} s_{j+1}]$, let w be any vertex in $(s_j \overrightarrow{P_1} s_{j+1})$. Since $N(w) \subset [s_j \overrightarrow{P_1} s_{j+1}]$ and G is triangle-free, we have $|V(P_1)| \geq |[s_j \overrightarrow{P_1} s_{j+1}]| \geq 2d(w) - 1 \geq 2\delta - 1$. ■

Theorem 1.7 is best possible in the sense that there exist bipartite graphs with minimum degree δ whose longest cycle has length precisely $4\delta - 4$.

In addition, these graphs have the property that longest cycles are not dominating cycles. For $\delta \geq 3$, simply take three or more disjoint copies of $K_{\delta,\delta-2}$ and join each vertex in the δ-sets to two new vertices u and v.

We also show that it is necessary to include the possibility that every longest cycle of G is a dominating cycle. Consider the complete bipartite graph $G = K_{p,q}$ where $q > p \geq 3$. This graph has $\delta = p$ and $c(G) = 2p < \min\{n, 4\delta - 5\}$. However, every longest cycle in such a graph is a dominating cycle.

4 Concluding Remarks

We conclude with a conjecture on k-connected triangle-free graphs.

Conjecture 4.1. *Let G be a k-connected triangle-free graph. Then $c(G) \geq 2\kappa(\delta - k + 1)$ or every longest cycle in G is a dominating cycle.*

Note that the cycle bound in Conjecture 4.1 differs from twice the bound in Conjecture 1.3 by $2k$. For $k = 2$, this is consistent with how our result for triangle-free graphs differs from twice the known result (Dirac's Theorem) for general graphs.

We also have examples to show that Conjecture 4.1, if true, is best possible in the same sense that Theorem 1.7 is best possible.

For $x \geq 1$, let G_j be the graph obtained by joining $k \geq 3$ independent vertices to the $j(k + x)$ vertices in the larger partite sets in $\bigcup_{i=1}^{j} K_{x,k+x}$. Then

- G_{k-1} does not achieve the cycle bound in Conjecture 4.1, but every longest cycle is dominating.
- G_k exactly achieves the cycle bound, and every longest cycle is dominating.
- G_{k+1} exactly achieves the cycle bound, but longest cycles in G_{k+1} are not dominating.

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