Network Augmentation and the Multigraph Conjecture

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Abstract

Let $\Gamma(n, m)$ denote the class of all graphs and multigraphs with n nodes and m edges. A central question in network reliability theory is the network augmentation problem: For $G \in \Gamma(n, m)$ fixed, what $H \in \Gamma(n, m + k)$ such that $G \subset H$ is t-optimal, that is, maximizes the tree number $t(H)$? In the network synthesis problem, where G is the empty graph on n vertices, it is conjectured that all t-optimal graphs are simple. We demonstrate that, in the general network augmentation case, there exists an infinite class of non-empty G for which the resulting t-optimal augmentations are multigraphs. This conclusion has ramifications for future attempts to prove or disprove the multigraph conjecture for the network synthesis problem.

1 Introduction

Graphs and multigraphs are often used to model communication networks, the nodes representing communication sites and the edges representing the communication links between the sites. When the links are not perfectly reliable the resulting model is a probabilistic graph whose individual edges appear with some probability. Under this model, a well-studied problem with natural applications is the following.

The Network Synthesis Problem *Given n nodes and m edges, each edge operating independently with probability p, characterize the graph that* *maximizes*

$$Rel(G) = \sum_{i=n-1}^{m} s_i(G)p^i(1-p)^{m-i}$$

where $s_i(G)$ denotes the number of spanning connected subgraphs of G that have exactly i edges.

The expression above, called the reliability polynomial of G, is the probability that the probabilistic graph is connected, or in network terms, the probability that any communication site can communicate with any other.

For some choices of n and m, the choice of optimal network topology depends on the value of p [**?**, **?**]. For many (n, m) pairs, however, there exists a certain graph G that maximizes $Rel(G)$ for any choice of p. Such graphs are called uniformly optimally reliable graphs, or more simply UOR graphs, and represent the best possible network design in their class regardless of edge probability p. It is NP-hard to identify UOR graphs [**?**], and to date only scattered classes of UOR classes are extant. Since identifying the graphs themselves has proved difficult, attention has focused on identifying features that such graphs must possess. All UOR graphs demonstrated to date are simple graphs, and a prominent conjecture is the following.

The Multigraph Conjecture *For $m \leq \binom{n}{2}$, any UOR graph must be simple.*

It is known that any UOR graph must also maximize $t(G)$, the number of trees of G, over all graphs in its class. Thus another related conjecture is that, for $m \leq \binom{n}{2}$, the graph with the most trees in its class must be simple.

Underlying the Multigraph Conjecture, at least intuitively, is the belief that the best edge additions to an existing network are always simple edges, or equivalently that any multigraph can be improved by "stripping out" the multiedges and resituating those removed edges in some simple manner. Accordingly, one approach to solving the Multigraph Conjecture is to consider another problem, also motivated by network modelling concerns, which is in some sense more general than the synthesis problem.

The Network Augmentation Problem *Given an existing probabilistic graph G, characterize the graph H which has k more edges than G and which maximizes the reliability polynomial subject to the condition $G \subset H$.*

In other words, if we have k edges to add to some base graph G, what is the best way to add them? We call such an addition of edges a k-edge augmentation of G. When our base graph is equal to the empty graph on n vertices this problem is in fact the network synthesis problem.

The purpose of this paper is to demonstrate that, for the general network augmentation problem, the intuition mentioned previously is false: there do exist simple graphs for which the t-optimal augmentation is a multigraph, even when other simple augmentations are available. This is demonstrated for the smallest augmentation case $k = 1$ in Section Two. In Section Three, one of the multigraph t-optimal augmentations from Section Two is shown to be uniformly more reliable than any simple augmentation as well. In Section Four we extend the results of Sections Two and Three to the 2-edge augmentation case. Finally in Section Five we demonstrate that for any choice of $k > 0$ there exist an infinite number of simple graphs G for which $G + ke$, that is, G with k copies of one edge e added, has more trees than any simple k-edge augmentation of G. This demonstrates that any k-edge uniformly most reliable augmentation of such a G, if it exists, must be a multigraph as well.

Our notation and terminology follows that in [**?**]. We emphasize that $G|e$ denotes the graph obtained from G by contracting the edge $e \in G$, or equivalently the graph obtained by identifying the end-vertices of $e \in G$. We also introduce two additional notations. Let $\Gamma(n, m)$ denote the class of all graphs and multigraphs which have n nodes and m edges. Finally, let $t_G(e)$ denote the number of trees of G which contain the edge $e \in G$.

2 Single-Edge Augmentations

The base graphs we will be augmenting all belong to a particular family of graphs or are slight modifications of that family, which we describe here. A star graph is a bipartite graph in which one of the parts is a single vertex, i.e. a star graph is $K_{1,s}$ for some s. Our base graphs will have complements consisting of two stars (and possibly some isolated vertices). Note that this implies that our base graphs and their simple edge augmentations will be members of the same family. An example of one of our base graphs is shown below. The graph pictured will turn out to be the smallest base graph known for which the best augmentation results in a multigraph.

Formulas for $t(G)$ for members of our family of graphs are available from a number of sources. One such statement is below.

Lemma 2.1 *[**?**] Let $\overline{G}$ consist of two stars with s and t leaves respectively, and possibly some isolated vertices. Then $t(G) = n^{n-2-s-t}(n-1)^{s+t-2}(n-1-s)(n-1-t)$.*

Since we will be trying to maximize $t(G)$ when $\overline{G}$ is of the form above, an important tool is the following.

Corollary 2.2 *Let $s+t=k$ be fixed, and let $\overline{G}$ consists of two stars of s and t leaves respectively. Then $t(G)$ is maximized when $|s-t|$ is made as small as possible, i.e. when the stars are as equal in order as possible. In particular, when k is of even parity we require $s=t$ and when k is odd we require $s=t-1$.*

PROOF: Rewrite $t(G)=n^{n-2-k}(n-1)^{k-2}\left[(n-1)^2-k(n-1)+st\right]$. The only quantity not fixed is st, which for $s+t$ fixed is maximized when $|s-t|$ is as small as possible. $\square$

Our multigraph augmentations will all involve the addition of multiple copies of the same edge. Call that edge $e \in G$. The following proposition will be of aid in evaluating $t(G+ke)$, the graph obtained by the addition of k copies of e.

Lemma 2.3 *Let G be a graph and let $e \in E(G)$. Then $t(G+ke)=t(G)+kt(G|e)$.*

PROOF: Recall that $t_G(e)$ denotes the number of trees of G that contain e. Each copy of e we add to G increases the number of trees by precisely $t_G(e)$, or $t(G+ke)=t(G)+kt_G(e)$. But $t_G(e)$ is equal to the number of trees of $G|e$, and the result follows. $\square$

Our $G+ke$ graphs can only be t-optimal and/or most reliable if we are adding additional copies of the edge of G that carries the most trees, i.e. if $t_G(e) \geq t_G(f)$ for all $f \neq e$. For graphs in our family this edge is easy to identify. We require one preliminary result.

Lemma 2.4 *[**?**] K_n is t-optimal.*

Lemma 2.5 *Let $G \in \Gamma(n,m)$ be such that $\overline{G}$ consists of exactly two stars (no isolated vertices). Then the edge of G which carries the most trees is the edge connecting the central nodes of the two stars.*

PROOF: We make use again of the fact that the number of trees an edge $e \in G$ carries is equal to $t(G|e)$. When e is the edge connecting the central nodes of the two stars, $G|e=K_{n-1}$. Since the complete graph is known to be t-optimal over all graphs and multigraphs in its class, and since all other edge contractions of G result in a graph in the class of K_{n-1}, e must carry at least as many trees as any other edge in G. $\square$

With the above tools in place the calculations are now straightforward. We will consider the n even and n odd case in turn.

Lemma 2.6 *Let $G \in \Gamma(n,m)$ with n even such that $\overline{G} = K_{1,\frac{n-2}{2}} \cup K_{1,\frac{n-2}{2}}$. Let e be the edge that connects the central nodes of the two stars. Then:*

$$t(G+e) = \frac{1}{4}n^2(n-1)^{n-4} + (n-1)^{n-3}$$

Let $G \in \Gamma(n,m)$ with n even such that $\overline{G} = K_{1,\frac{n-2}{2}} \cup K_{1,\frac{n-2}{2}-1} \cup K_1$. Then:

$$t(G) = \frac{1}{4}n^2(n-1)^{n-5}(n+2)$$

Proof: The first expression for $G+e$ is derived by using Lemma 2.3 to obtain $t(G+e) = t(G) + t(G|e) = t(G) + t(K_{n-1})$ and substituting the appropriate expressions for s and t in Lemma 2.1. The second expression is a direct application of Lemma 2.1. □

Theorem 2.7 *Let $G \in \Gamma(n,m)$ with n even such that $\overline{G} = K_{1,\frac{n-2}{2}} \cup K_{1,\frac{n-2}{2}}$. Let e be the edge that connects the central nodes of the two stars. Then for $n \geq 8$ the t-optimal 1-edge augmentation of G consists of G with an additional e edge added.*

Proof: The only simple edge additions create a graph of the form described in Lemma 2.1. Furthermore, the simple edge augmentations are identical up to isomorphism. Call this graph G_s. By Lemma 2.5 we know the best multigraph augmentation involves adding an additional e edge. Call this graph G_m. Using Lemma 2.6 and simplifying gives

$$\begin{aligned} t(G_m) - t(G_s) &= \frac{1}{4}n^2(n-1)^{n-4} + (n-1)^{n-3} \\ &\quad -\frac{1}{4}n^2(n-1)^{n-5}(n+2) \\ &= \frac{1}{4}(n-1)^{n-5}(n^2 - 8n + 4) \end{aligned}$$

It is easily verified that $n^2 - 8n + 4$ is positive for $n \geq 8$ and negative for other positive n, and our result is proved. □

The n odd case is very similar. Because of the asymmetry between the stars there are now two possible graphs obtainable via adding a simple edge, but by Corollary 2.2 we need only consider the simple edge addition that results in $\overline{G+e}$ consisting of two stars with identical numbers of leaves and an isolated vertex.

Lemma 2.8 *Let $G \in \Gamma(n,m)$ with n odd such that $\overline{G} = K_{1,\frac{n-1}{2}} \cup K_{1,\frac{n-3}{2}}$. Let e be the edge that connects the central nodes of the two stars. Then:*

$$t(G+e) = \frac{1}{4}(n+1)(n-1)^{n-3} + (n-1)^{n-3}$$

Let $G_s \in \Gamma(n,m)$ with n odd such that $\overline{G} = K_{1,\frac{n-3}{2}} \cup K_{1,\frac{n-3}{2}} \cup K_1$. Then:

$$t(G_s) = \frac{1}{4}n(n-1)^{n-5}(n+1)^2$$

Proof: Exactly as in the even case. □

The odd result now follows.

Theorem 2.9 *Let $H \in \Gamma(n,m)$ with n odd such that $\overline{H} = K_{1,\frac{n-1}{2}} \cup K_{1,\frac{n-3}{2}}$. Let e be the edge that connects the central nodes of the two stars. Then for $n \geq 11$ the $G \in \Gamma(n,m+1)$ with $H \subset G$ that has the most trees consists of H with another edge e added.*

Proof: The best simple edge addition creates a graph of the form described in Lemma 2.8. Accordingly, call this graph G_s and let the graph created by multigraphing e be denoted G_m. By Lemma 2.8 we have

$$\begin{aligned} t(G_m) - t(G_s) &= \frac{1}{4}(n+1)(n-1)^{n-3} + (n-1)^{n-3} \\ &\quad -\frac{1}{4}n(n-1)^{n-5}(n+1)^2 \\ &= \frac{1}{4}(n-1)^{n-5}(n^2-10n+5) \end{aligned}$$

It is easily verified that $n^2 - 10n + 5$ is positive for $n \geq 11$ and negative for other positive n, and our result is proved. □

In summary we list the cases where the t-optimal single-edge augmentation results in a multigraph.

Theorem 2.10 *Let G be the graph on n vertices whose complement $\overline{G} = K_{1,s} \cup K_{1,t}$ where $s = \lfloor \frac{n-2}{2} \rfloor$ and $t = \lceil \frac{n-2}{2} \rceil$. Let e be the edge that connects the central nodes of the two stars. Then if $n = 8$ or $n \geq 10$, adding an additional multiedge e creates more spanning trees than adding any single simple edge.*

We note that none of the graphs here examined, base graphs or their single-edge augmentations, are t-optimal graphs in the synthesis sense. (It is known that K_n minus a matching is t-optimal over all simple graphs in its class, and all of the graphs examined here have fewer trees than K_n minus a matching.) We also note that, for the graph examined, the t-optimal 2-edge augmentation is simple, and other k-edge augmentations for $3 \leq k \leq 6$ are as well.

3 Reliable Single-Edge Augmentations

It is also possible to show that, for the graph pictured in Figure One, the multigraph single-edge augmentation is uniformly more reliable than the simple single-edge augmentation. We require two preliminary facts, the first of which is widely known and the second of which is a consequence of the definition of the reliability polynomial.

Lemma 3.1 ***The Factor Theorem*** *[?] For any graph or multigraph G, including those with loops,*

$$Rel(G) = pRel(G|e) + (1-p)Rel(G-e)$$

Lemma 3.2 *If $e \in G$ is a loop, then $Rel(G) = Rel(G-e)$.*

Proof: Recall that $Rel(G)$ is just the probability that the probabilistic graph is connected. Whether e is operational or not has no bearing on whether G is connected or not, and the result follows. $\square$

We are now prepared to show that the multigraph augmentation previously shown to be t-optimal is in fact uniformly more reliable than the simple edge augmentation available. The proof is computational and aided by the relatively small size of the graph.

Theorem 3.3 *Let $G \in \Gamma(8,22)$ be the graph pictured in Figure One, and let G_s and G_m be the simple and multigraph single-edge augmentations previously discussed for this case. Then $Rel(G_s) < Rel(G_m)$ for all $p \in (0,1)$.*

Proof: We calculate the reliability polynomials directly. Using Maple 8.0, we can calculate the reliability polynomials for simple graphs. The calculation follows.

$$Rel(G_s) \quad = \quad p^7 \left(2160p^{16} - 40080p^{15} + 349992p^{14} - 1909932p^{13}\right.$$

$$+7293360p^{12} - 20675392p^{11} + 45034474p^{10} - 76933841p^{9}$$
$$+104254382p^{8} - 112541819p^{7} + 96560945p^{6} - 65242955p^{5}$$
$$+34085708p^{4} - 13337485p^{3} + 3695483p^{2} - 649879p$$
$$+54880)$$

G_m can be decomposed into simple graphs using the Factor Theorem, and is thus amenable to calculation as well. Let e denote either one of the multiedges.

$$\begin{aligned}
Rel(G_m) &= pRel(G_m|e) + (1-p)Rel(G_m - e) \\
&= pRel(K_7) + (1-p)Rel(G) \\
&= p^7 \left(1800p^{16} - 33912p^{15} + 300564p^{14} - 1664130p^{13}\right. \\
&\quad +6444918p^{12} - 18521964p^{11} + 40882587p^{10} - 70742433p^{9} \\
&\quad +97057111p^{8} - 106025135p^{7} + 92010526p^{6} - 62845533p^{5} \\
&\quad +33171204p^{4} - 13104705p^{3} + 3663267p^{2} - 649387p \\
&\quad \left.+55223\right)
\end{aligned}$$

On the second line above we have taken advantage of Lemma 3.2. ($G_m|e$ is in fact K_7 with a single loop attached.) Subtracting these two expressions and simplifying gives

$$\begin{aligned}
Rel(G_m) - Rel(G_s) &= -p^7 \left(360p^{12} - 4728p^{11} + 28356p^{10} - 102570p^{9}\right. \\
&\quad +248754p^{8} - 424840p^{7} + 521367p^{6} - 459314p^{5} \\
&\quad +283699p^{4} - 115696p^{3} + 26818p^{2} - 1864p \\
&\quad \left.-343\right)(1-p)^4
\end{aligned}$$

It is verifiable (via Sturm sequences, for example), that the expression above is positive for all $p \in (0,1)$, and so $Rel(G_m) > Rel(G_s)$ for all $p \in (0,1)$. $\square$

Thus multigraph augmentations may be, not only t-optimal when compared to simple augmentations, but uniformly more reliable as well. It may very well be that a large number of the family of graphs under discussion are uniformly more reliable than the simple augmentation alternatives, but we know of no analytic method that would demonstrate that fact.

4 Two-Edge Augmentations

Our base graphs are slight alterations of the family of graphs previously discussed. For this new family we take the graphs from Section Two and omit the edge e which connects the stars. An example which corresponds to the previous figure is shown below.

The analysis follows closely the lines of the previous section, and consequently where appropriate we omit some of the proofs.

Theorem 4.1 *[**?**] Let $\overline{G}$ consist of two stars with s and t leaves respectively, as well as the edge connecting the central nodes of the two stars, and possibly some isolated vertices. Then $t(G) = n^{n-2-s-t}(n-1)^{s+t-2}(n-1-s)(n-1-t) - n^{n-3-s-t}(n-1)^{s+t-1}(2n-2-s-t)$.*

Corollary 4.2 *Let $s + t = k$ be fixed, and let $\overline{G}$ be of the form described above. Then $t(G)$ is maximized when $|s - t|$ is made as small as possible, i.e. when the stars are as equal in size as possible. In particular, when k is of even parity we require $s = t$ and when k is odd we require $s = t - 1$.*

We again consider the n even and n odd case in turn. From the discussions of the previous section it is easy to see that the following holds.

Lemma 4.3 *Let $G \in \Gamma(n, m)$ with n even such that $\overline{G} = K_{1,\frac{n-2}{2}} \cup K_{1,\frac{n-2}{2}} \cup e$, where e is the edge joining the central nodes of the stars. Let $G + 2e = G_m$. Then*

$$t(G_m) \quad = \quad \frac{1}{4}n^2(n-1)^{n-4} - (n-1)^{n-3}$$

Before we compare the t-optimal simple augmentation to the t-optimal multigraph augmentation we need to determine which simple augmentation is best. There are two simple augmentations which need to be compared, one of which adds edge e and one of which does not. Up to isomorphism, there is but one simple way to add 2 edges to our base graph while adding edge e. $t(G)$ has been derived for this case in the previous section in Lemma 2.6, and as before we will call this graph G_s. There are two ways to add two simple non-e edges to our base graph, but by Corollary 4.2 we need only consider the case where the two edges are distributed evenly, one to each star. We will call this simple graph G_t.

Lemma 4.4 *Let $G_t \in \Gamma(n, m)$ with n even such that $\overline{G_t}$ consists of $K_{1,\frac{n-4}{2}} \cup K_{1,\frac{n-4}{2}} \cup e$ and two isolated vertices, where e is the edge joining the central nodes of the stars. Then:*

$$t(G_t) = \frac{1}{4}n(n-1)^{n-6}(n^3+8)$$

Lemma 4.5 *Let $G_s \in \Gamma(n, m)$ with n even such that $\overline{G_s}$ consists of $K_{1,\frac{n-2}{2}} \cup K_{1,\frac{n-4}{2}}$, and let $G_t \in \Gamma(n, m)$ with n even such that $\overline{G_t}$ consists of $K_{1,\frac{n-2}{2}} \cup K_{1,\frac{n-4}{2}} \cup e$ where e is the edge joining the central nodes of the two stars. Then $t(G_s) > t(G_t)$.*

Proof: The expression for $t(G_s)$ was derived in the previous section and the expression for $t(G_t)$ immediately preceded this lemma. Subtracting we arrive at:

$$\begin{aligned} t(G_s) - t(G_t) &= \frac{1}{4}n^2(n-1)^{n-4} + (n-1)^{n-3} - \frac{1}{4}n(n-1)^{n-6}(n^3+8) \\ &= \frac{1}{4}(n-1)^{n-6}(2n^3 - 11n^2 + 12n - 12) \end{aligned}$$

which is positive for all $n \geq 5$. Since the definitions of the complement require $n \geq 6$, the result follows. □

Since $t(G_s) > t(G_t)$ we need only consider graphs of type G_s when making our comparison to multigraph augmentations. However, those graphs are exactly the G_s graphs that were compared to our G_m in the previous section. Since G_m was proven to be t-optimal when compared to G_s previously, we have the following result.

Lemma 4.6 *Let $G \in \Gamma(n,m)$ with n even such that $\overline{G}$ consists of $K_{1,\frac{n-2}{2}} \cup K_{1,\frac{n-2}{2}} \cup e$ where e is the edge joining the central nodes of the two stars. Then for $n \geq 8$ the t-optimal 2-edge augmentation of G is a multigraph.*

For completeness we turn now to the n odd case. By now the methods used to derive the results are familiar, and we state the result without proof.

Theorem 4.7 *Let $G \in \Gamma(n,m)$ with n odd such that $\overline{G} = K_{1,\frac{n-1}{2}} \cup K_{1,\frac{n-3}{2}} \cup e$, where e is the edge joining the central nodes of the stars. Then for $n \geq 11$ the t-optimal 2-edge augmentation of G is a multigraph.*

In summary we state the following result for 2-edge augmentations.

Theorem 4.8 *Let G be the graph on n vertices whose complement $\overline{G} = K_{1,s} \cup K_{1,t} \cup e$ where $s = \lfloor \frac{n-2}{2} \rfloor$ and $t = \lceil \frac{n-2}{2} \rceil$ and e is the edge joining the central nodes of the stars. Then if $n = 8$ or $n \geq 10$, the t-optimal 2-edge augmentation of G is a multigraph.*

For the base graph pictured in Figure Two, it is possible to show that the multigraph 2-edge augmentation G_m discussed in this section is in fact also uniformly more reliable than any of the available simple 2-edge augmentations. That analysis can be done computationally along the lines of Section Two and is omitted here. The result is as follows.

Theorem 4.9 *Let $G \in \Gamma(8,21)$ be the graph pictured in Figure Two, and let G_m be the multigraph 2-edge augmentation previously discussed for this case. Then for any simple 2-edge augmentation G_s, we have $Rel(G_s) < Rel(G_m)$ for all $p \in (0,1)$.*

Thus for the graph shown the 2-edge multigraph augmentation is uniformly more reliable than any simple augmentation.

5 Larger Edge Augmentations

We now demonstrate that, for any k-edge augmentation considered, there exist an infinite number of simple graphs G for which $t(G + ke) > t(G_s)$, where G_s is any simple edge augmentation.

Theorem 5.1 *Let G be a graph with an even number of nodes whose complement consists of two identical stars of $\frac{n-2}{2}$ leaves each, and let e denote the edge connecting the central nodes of the two stars. Let G_s denote any simple $2k$ edge augmentation of G and let $G_m = G{+}2ke$. Then if $n > 8k{+}6$, we have $t(G_m) > t(G_s)$.*

Proof: By a previous theorem we need only consider the simple case in which $\overline{G_s}$ consists of two identical stars with $\frac{n-2}{2} - k$ leaves each. We are thus comparing the two quantities

$$\begin{aligned} t(G_s) &= \frac{1}{4}n^{2k}(n-1)^{n-4-2k}(n+2k)^2 \\ t(G_m) &= \frac{1}{4}n^2(n-1)^{n-4} + 2k(n-1)^{n-3} \\ &= \frac{1}{4}(n-1)^{n-4}(n^2 + 8kn - 8k) \end{aligned}$$

A little algebra reveals that we have $t(G_m) > t(G_s)$ if and only if

$$\left(1 - \frac{1}{n}\right)^{2k} > 1 - \frac{4kn - 12k}{n^2 + 8kn - 8k}$$

It is apparent that both fractions involved have magnitudes smaller than one, and thus taking logs of both sides and expanding into power series is justified. Doing so gives us the equivalent condition

$$-2k\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{1}{n}\right)^i > -\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{4kn - 12k}{n^2 + 8kn - 8k}\right)^i$$

or

$$\sum_{i=1}^{\infty}\frac{1}{i}\left(\frac{1}{n}\right)^i < \sum_{i=1}^{\infty}\frac{(2k)^{i-1}}{i}\left(\frac{2n - 6}{n^2 + 8kn - 8k}\right)^i$$

This condition is obviously satisfied if

$$\frac{2n-6}{n^2 + 8kn - 8k} > \frac{1}{n}$$

for all n, k chosen. Cross-multiplying and combining terms yields

$$n^2 - (8k+6)n + 8k > 0$$

a quadratic in n. Thus the condition above holds if n is larger than the largest root of the quadratic. This root is positive and smaller than $8k+6$, and our result follows. $\square$

For the odd case a similar result holds.

Theorem 5.2 *Let G be a graph on an odd number of nodes whose complement consists of two stars of $\frac{n-1}{2}$ leaves and $\frac{n-3}{2}$ leaves each, and let e denote the edge connecting the central nodes of the two stars. Let G_s denote any simple $2k+1$ edge augmentation of G and let $G_m = G + (2k+1)e$. Then if $n > 10k + 10$, we have $t(G_m) > t(G_s)$.*

Proof: By a previous theorem we need only consider the simple case in which $\overline{G_s}$ consists of two identical stars with $\frac{n-3}{2} - k$ leaves each. We are thus comparing the two quantities

$$\begin{aligned}
t(G_s) &= \frac{1}{4} n^{2k+1}(n-1)^{n-5-2k}(n+2k+1)^2 \\
t(G_m) &= \frac{1}{4}(n+1)(n-1)^{n-3} + (2k+1)(n-1)^{n-3} \\
&= \frac{1}{4}(n-1)^{n-3}(n^2 + (8k+4)n - 8k - 5)
\end{aligned}$$

A little algebra reveals that we have $t(G_m) > t(G_s)$ if and only if

$$\left(1 - \frac{1}{n}\right)^{2k+1} > 1 - \frac{(4k+2)n - (4k^2 + 12k + 6)}{n^2 + (8k+4)n - 8k - 5}$$

It is apparent that both fractions involved have magnitudes smaller than one, and thus taking logs of both sides and expanding into power series is justified. Doing so gives us the equivalent condition

$$-(2k+1)\sum_{i=1}^{\infty} \frac{1}{i}\left(\frac{1}{n}\right)^i > -\sum_{i=1}^{\infty} \frac{1}{i}\left(\frac{(4k+2)n - (4k^2+12k+6)}{n^2 + (8k+4)n - 8k - 5}\right)^i$$

or

$$\sum_{i=1}^{\infty} \frac{1}{i}\left(\frac{1}{n}\right)^i < \sum_{i=1}^{\infty} \frac{(2k+1)^{i-1}}{i}\left(\frac{2n - (2k+5+\frac{1}{2k+1})}{n^2 + (8k+4)n - 8k - 5}\right)^i$$

This condition is obviously satisfied if

$$\frac{2n - (2k+5+\frac{1}{2k+1})}{n^2 + (8k+4)n - 8k - 5} > \frac{1}{n}$$

for all n, k chosen. Cross-multiplying and combining terms yields

$$n^2 - \left(10k + 9 + \frac{1}{2k+1}\right) n + 8k + 5 > 0$$

a quadratic in n. Thus the condition above holds if n is larger than the largest root of the quadratic. This root is positive and smaller than the largest root of

$$n^2 - (10k + 10)n + 8k + 5 > 0$$

The largest root of the above quadratic is smaller than $10k + 10$, and the result follows. □

In particular we have shown the following to be true.

Theorem 5.3 *For any $k > 0$, there exists an infinite number of graphs G with an $e \in G$ such that $t(G + ke) > t(G_s)$ for any simple k-edge augmentation G_s. In particular, the t-optimal k-edge augmentation of G is a multigraph, and any k-edge uniformly most reliable augmentation for G, if it exists, must be a multigraph.*

It is worth noting that the above does not prove that t-optimal augmentations (or most reliable augmentations) will necessarily result in arbitrarily large edge multiplicities. In fact, in the cases we have examined computationally, all t-optimal augmentations have resulted in at most an edge multiplicity of 2.

6 Consequences and Future Directions

The vast majority of research on network synthesis and augmentation problems begins with the explicit assumption that the optimal graph topologies are simple. The graphs demonstrated here show that this assumption depends heavily on the initial network considered. Moreover, when taken together with the result of Myrvold, et al. [**?**], which demonstrated a multigraph which is optimal over planar networks, these results suggest that multigraphs may have more to offer than previously thought in the area of network optimization.

The graphs discussed here also impact future efforts to prove or disprove the Multigraph Conjecture for the network synthesis problem. Any attempted proof of the claim that all t-optimal networks are simple cannot be based on the intuitively appealing process of moving multiedge(s) to some other simple location. For the multigraph augmentations of the graphs of Figures One and Two, there are no better simple locations.

Lastly, we state two conjectures and a problem based on features of the graphs seen.

Conjecture *For any $k > 0$, there exist an infinite number of graphs G such that a uniformly most reliable k-edge augmentation of G exists and is a multigraph.*

Conjecture *For $m + k \leq \binom{n}{2}$, no t-optimal (or uniformly most reliable) augmentation results in an edge with multiplicity 3 or greater.*

Recall that t-optimal synthesis graphs are t-optimal over all possible $G \in \Gamma(n, m)$, i.e. there are no subgraph conditions as in the augmentation problem.

Problem *Let P be a property of t-optimal synthesis graphs. Determine $k_{n,m}$ such that, if $k \geq k_{n,m}$, any t-optimal k-edge augmentation of a simple graph $G \in \Gamma(n, m)$ must have P as well.*

Since all optimal k-edge augmentations for $k = \binom{n}{2} - m$ must result in the complete graph K_n, the challenge is to determine if a smaller value holds for $k_{n,m}$ and if so, what the bound is. For example, for the base graph $G \in \Gamma(8, 22)$ pictured in Figure One, any t-optimal edge augmentation of 2 or more edges results in a simple graph. Is this true for all $G \in \Gamma(8, 22)$? If so, then $k_{8,22} = 2$ for the property of being simple.

References