Degree Sequences and the Existence of k-Factors

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Abstract

We consider sufficient conditions for a degree sequence π to be forcibly k-factor graphical. We note that previous work on degrees and factors has focused primarily on finding conditions for a degree sequence to be potentially k-factor graphical.

We first give a theorem for π to be forcibly 1-factor graphical and, more generally, forcibly graphical with deficiency at most $\beta \geq 0$. These theorems are equal in strength to Chvátal's well-known hamiltonian theorem, i.e., the best monotone degree condition for hamiltonicity. We then give an equally strong theorem for π to be forcibly 2-factor graphical. Unfortunately, the number of nonredundant conditions that must be checked increases significantly in moving from $k = 1$ to $k = 2$, and we conjecture that the number of nonredundant conditions in a best monotone theorem for a k-factor will increase superpolynomially in k.

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This suggests the desirability of finding a theorem for π to be forcibly k-factor graphical whose algorithmic complexity grows more slowly. In the final section, we present such a theorem for any $k \geq 2$, based on Tutte's well-known factor theorem. While this theorem is not best monotone, we show that it is nevertheless tight in a precise way, and give examples illustrating this tightness.

Keywords: k-factor of a graph, degree sequence, best monotone condition

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1 Introduction

We consider only undirected graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms or notation is [4].

A *degree sequence* of a graph on n vertices is any sequence $\pi = (d_1, d_2, \ldots, d_n)$ consisting of the vertex degrees of the graph. In contrast to [4], we will usually assume the sequence is in nondecreasing order. We generally use the standard abbreviated notation for degree sequences, e.g., $(4, 4, 4, 4, 4, 5, 5)$ will be denoted $4^5 5^2$. A sequence of integers $\pi = (d_1, d_2, \ldots, d_n)$ is called *graphical* if there exists a graph G having π as one of its degree sequences, in which case we call G a *realization* of π. If $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ are two integer sequences, we say π' *majorizes* π, denoted $\pi' \geq \pi$, if $d'_j \geq d_j$ for $1 \leq j \leq n$. If P is a graphical property (e.g., k-connected, hamiltonian), we call a graphical degree sequence *forcibly* (respectively, *potentially*) P *graphical* if every (respectively, some) realization of π has property P.

Historically, the degree sequence of a graph has been used to provide sufficient conditions for a graph to have a certain property, such as k-connected or hamiltonian. Sufficient conditions for a degree sequence to be forcibly hamiltonian were given by several authors, culminating in the following theorem of Chvátal [6] in 1972.

Theorem 1.1 ([6]). *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical degree sequence, with $n \geq 3$. If $d_i \leq i < \frac{1}{2}n$ implies $d_{n-i} \geq n - i$, then π is forcibly hamiltonian graphical.*

Unlike its predecessors, Chvátal's theorem has the property that if it does not guarantee that a graphical degree sequence π is forcibly hamiltonian graphical, then π is majorized by some degree sequence π' which has a nonhamiltonian realization. As we'll see, this fact implies that Chvátal's theorem is the strongest of an entire class of theorems giving sufficient conditions for π to be forcibly hamiltonian graphical.

A *factor* of a graph G is a spanning subgraph of G. A *k-factor* of G is a factor whose vertex degrees are identically k. For a recent survey on graph factors, see [14]. In the present paper, we develop sufficient conditions for a degree sequence to be forcibly k-factor graphical. We note that previous work relating degrees and the existence of factors has focused primarily on sufficient conditions for π to be potentially k-factor graphical. The following obvious necessary condition was conjectured to be sufficient by Rao and Rao [15], and this was later proved by Kundu [11].

Theorem 1.2 ([11])**.** *The sequence $\pi = (d_1, d_2, \ldots, d_n)$ is potentially k-factor graphical if and only if*

(1) $(d_1, d_2, \ldots, d_n)$ *is graphical, and*

(2) $(d_1 - k, d_2 - k, \ldots, d_n - k)$ *is graphical.*

Kleitman and Wang [9] later gave a proof of Theorem 1.2 that yielded a polynomial algorithm constructing a realization G of π with a k-factor. Lovász [13] subsequently gave a very short proof of Theorem 1.2 for the special case $k = 1$, and Chen [5] produced a short proof for all $k \geq 1$.

In Section 2, we give a theorem for π to be forcibly graphical with deficiency at most β (i.e., have a matching missing at most β vertices), and show this theorem is strongest in the same sense as Chvátal's hamiltonian degree theorem. The case $\beta = 0$ gives the strongest result for π to be forcibly 1-factor graphical. In Section 3, we give the strongest theorem, in the same sense as Chvátal, for π to be forcibly 2-factor graphical. But the increase in the number of nonredundant conditions which must be checked as we move from a 1-factor to a 2-factor is notable, and we conjecture the number of such conditions in the best monotone theorem for π to be forcibly k-factor graphical increases superpolynomially in k. Thus it would be desirable to find a theorem for π to be forcibly k-factor graphical in which the number of nonredundant conditions grows in a more reasonable way. In Section 4, we give such a theorem for $k \geq 2$, based on Tutte's well-known factor theorem. While our theorem is not best monotone, it is nevertheless tight in a precise way, and we provide examples to illustrate this tightness.

We conclude this introduction with some concepts which are needed in the sequel. Let P denote a graph property (e.g., hamiltonian, contains a k-factor, etc.) such that whenever a spanning subgraph of G has P, so does G. A function $f : \{\text{Graphical Degree Sequences}\} \to \{0, 1\}$ such that $f(\pi) = 1$ implies π is forcibly P graphical, and $f(\pi) = 0$ implies nothing in this regard, is called a *forcibly P function*. Such a function is called *monotone* if $\pi' \geq \pi$ and $f(\pi) = 1$ implies $f(\pi') = 1$, and *weakly optimal* if $f(\pi) = 0$ implies there exists a graphical sequence $\pi' \geq \pi$ such that π' has a realization G' without P. A forcibly P function which is both monotone and weakly optimal is the best monotone forcibly P function, in the following sense.

Theorem 1.3. *If f, f_0 are monotone, forcibly P functions, and f_0 is weakly optimal, then $f_0(\pi) \geq f(\pi)$, for every graphical sequence π.*

Proof: Suppose to the contrary that for some graphical sequence π we have $1 = f(\pi) > f_0(\pi) = 0$. Since f_0 is weakly optimal, there exists a graphical sequence $\pi' \geq \pi$ such that π' has a realization G' without P, and thus $f(\pi') = 0$. But $\pi' \geq \pi$, $f(\pi) = 1$ and $f(\pi') = 0$ imply f cannot be monotone, a contradiction. ■

A theorem T giving a sufficient condition for π to be forcibly P corresponds to the forcibly P function f_T given by: $f_T(\pi) = 1$ if and only if T implies π is forcibly P. It is well-known that if T is Theorem 1.1 (Chvátal's theorem), then f_T is both monotone and weakly optimal, and thus the best monotone forcibly hamiltonian function in the above sense. In the sequel, we will simplify the formally correct 'f_T is monotone, etc.' to 'T is monotone, etc..'

2 Best monotone condition for a 1-factor

In this section we present best monotone conditions for a graph to have a large matching. These results were first obtained by Las Vergnas [12], and can also be obtained from results in Bondy and Chvátal [3]. For the convenience of the reader, we include the statement of the results and short proofs below.

The *deficiency* of G, denoted $\mathrm{def}(G)$, is the number of vertices unmatched under a maximum matching in G. In particular, G contains a 1-factor if and only if $\mathrm{def}(G) = 0$.

We first give a best monotone condition for π to be forcibly graphical with deficiency at most β, for any $\beta \geq 0$.

Theorem 2.1 ([3, 12])**.** *Let G have degree sequence $\pi = (d_1 \leq \cdots \leq d_n)$, and let $0 \leq \beta \leq n$ with $\beta \equiv n \pmod 2$. If*

$$d_{i+1} \leq i - \beta < \tfrac{1}{2}(n - \beta - 1) \implies d_{n+\beta-i} \geq n - i - 1,$$

then $\mathrm{def}(G) \leq \beta$*.*

The condition in Theorem 2.1 is clearly monotone. Furthermore, if π does not satisfy the condition for some $i \geq \beta$, then π is majorized by $\pi' = (i - \beta)^{i+1} (n - i - 2)^{n-2i+\beta-1}(n-1)^{i-\beta}$. But π' is realizable as $K_{i-\beta} + (\overline{K_{i+1}} \cup K_{n-2i+\beta-1})$, which has deficiency $\beta + 2$. Thus Theorem 2.1 is weakly optimal, and the condition of the theorem is best monotone.

Proof of Theorem 2.1: Suppose π satisfies the condition in Theorem 2.1, but $\mathrm{def}(G) \geq \beta + 2$. (The condition $\beta \equiv n \pmod 2$ guarantees that $\mathrm{def}(G) - \beta$ is always even.) Define $G' \doteq K_{\beta+1} + G$, with degree sequence $\pi' = (d_1 + \beta + 1, \ldots, d_n + \beta + 1, ((n-1) + \beta + 1)^{\beta+1})$. Note that the number of vertices of G' is odd.

Suppose G' has a Hamilton cycle. Then, by taking alternating edges on that cycle, there is a matching covering all vertices of G' except one vertex, and we can choose that missed vertex freely. So choose a matching covering all but one of the $\beta + 1$ new vertices. Removing the other β new vertices as well, the remaining edges form a matching covering all but at most β vertices from G, a contradiction.

Hence G' cannot have a Hamilton cycle, and π' cannot satisfy the condition in Theorem 1.1. Thus there is some $i \geq \beta + 1$ such that

$$d_i + \beta + 1 \leq i < \tfrac{1}{2}(n + \beta + 1) \qquad \text{and} \qquad d_{n+\beta+1-i} + \beta + 1 \leq (n + \beta + 1) - i - 1.$$

Subtracting $\beta+1$ throughout this equation gives

$$d_i \le i-\beta-1 < \tfrac{1}{2}(n-\beta-1) \qquad \text{and} \qquad d_{n+\beta+1-i} \le n-i-1.$$

Replacing i by $j+1$ we get

$$d_{j+1} \le j-\beta < \tfrac{1}{2}(n-\beta-1) \qquad \text{and} \qquad d_{n+\beta-j} \le n-j-2.$$

Thus π fails to satisfy the condition in Theorem 2.1, a contradiction. ■

As an important special case, we give the best monotone condition for a graph to have a 1-factor.

Corollary 2.2 ([3, 12])**.** *Let G have degree sequence $\pi = (d_1 \le \cdots \le d_n)$, with $n \ge 2$ and n even. If*

$$d_{i+1} \le i < \tfrac{1}{2}n \implies d_{n-i} \ge n-i-1, \tag{1}$$

then G contains a 1-factor.

We note in passing that (1) is Chvátal's best monotone condition for G to have a hamiltonian path [6].

3 Best monotone condition for a 2-factor

We now give a best monotone condition for the existence of a 2-factor. In what follows we abuse the notation by setting $d_0 = 0$.

Theorem 3.1. *Let G have degree sequence $\pi = (d_1 \le \cdots \le d_n)$, with $n \ge 3$. If*

(i) n *odd* $\implies d_{(n+1)/2} \ge \frac{1}{2}(n+1)$;

(ii) n *even* $\implies d_{(n-2)/2} \ge \frac{1}{2}n$ *or* $d_{(n+2)/2} \ge \frac{1}{2}(n+2)$;

(iii) $d_i \le i$ *and* $d_{i+1} \le i+1 \implies d_{n-i-1} \ge n-i-1$ *or* $d_{n-i} \ge n-i$, *for* $0 \le i \le \frac{1}{2}(n-2)$;

(iv) $d_{i-1} \le i$ *and* $d_{i+2} \le i+1 \implies d_{n-i-3} \ge n-i-2$ *or* $d_{n-i} \ge n-i-1$, *for* $1 \le i \le \frac{1}{2}(n-5)$,

then G contains a 2-factor.

The condition in Theorem 3.1 is easily seen to be monotone. Furthermore, if π fails to satisfy any of (i) through (iv), then π is majorized by some π' having a realization G' without a 2-factor. In particular, note that

- if (i) fails, then π is majorized by $\pi' = \left(\frac{1}{2}(n-1)\right)^{(n+1)/2}(n-1)^{(n-1)/2}$, having realization $K_{(n-1)/2} + \overline{K_{(n+1)/2}}$;
- if (ii) fails, then π is majorized by $\pi' = \left(\frac{1}{2}(n-2)\right)^{(n-2)/2}\left(\frac{1}{2}n\right)^2(n-1)^{(n-2)/2}$, having realization $K_{(n-2)/2} + (\overline{K_{(n-2)/2}} \cup K_2)$;

- if (iii) fails for some i, then π is majorized by $\pi' = i^i(i+1)^1(n-i-2)^{n-2i-2}$ $(n-i-1)^1(n-1)^i$, having realization $K_i + (\overline{K_{i+1}} \cup K_{n-2i-1})$ together with an edge joining $\overline{K_{i+1}}$ and K_{n-2i-1};
- if (iv) fails for some i, then π is majorized by $\pi' = i^{i-1}(i+1)^3(n-i-3)^{n-2i-5}$ $(n-i-2)^3(n-1)^i$, having realization $K_i + (\overline{K_{i+2}} \cup K_{n-2i-2})$ together with three independent edges joining $\overline{K_{i+2}}$ and K_{n-2i-2}.

It is immediate that none of the above realizations contain a 2-factor. Hence, Theorem 3.1 is weakly optimal, and the condition of the theorem is best monotone.

Proof of Theorem 3.1: Suppose π satisfies (i) through (iv), but G has no 2-factor. We may assume the addition of any missing edge to G creates a 2-factor. Let $v_1, \ldots, v_n$ be the vertices of G, with respective degrees $d_1 \leq \cdots \leq d_n$, and assume v_j, v_k are a nonadjacent pair with $j+k$ as large as possible, and $d_j \leq d_k$. Then v_j must be adjacent to $v_{k+1}, v_{k+2}, \ldots, v_n$ and so

$$d_j \geq n-k. \tag{2}$$

Similarly, v_k must be adjacent to $v_{j+1}, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$, and so

$$d_k \geq n-j-1. \tag{3}$$

Since $G + (v_j, v_k)$ has a 2-factor, G has a spanning subgraph consisting of a path P joining v_j and v_k, and $t \geq 0$ cycles $C_1, \ldots, C_t$, all vertex disjoint.

We may also assume v_j, v_k and P are chosen such that if v, w are any nonadjacent vertices with $d_G(v) = d_j$ and $d_G(w) = d_k$, and if P' is any (v,w)-path such that $G - V(P')$ has a 2-factor, then $|P'| \leq |P|$. Otherwise, re-index the set of vertices of degree d_j (resp., d_k) so that v (resp., w) is given the highest index in the set.

Since G has no 2-factor, we cannot have independent edges between $\{v_j, v_k\}$ and two consecutive vertices on any of the C_μ, $0 \leq \mu \leq t$. Similarly, we cannot have $d_P(v_j) + d_P(v_k) \geq |V(P)|$, since otherwise $\langle V(P) \rangle$ is hamiltonian and G contains a 2-factor. This means

$$\begin{gathered} d_{C_\mu}(v_j) + d_{C_\mu}(v_k) \leq |V(C_\mu)| \quad \text{for } 0 \leq \mu \leq t, \\ \text{and} \quad d_P(v_j) + d_P(v_k) \leq |V(P)| - 1. \end{gathered} \tag{4}$$

It follows immediately that

$$d_j + d_k \leq n-1. \tag{5}$$

We distinguish two cases for $d_j + d_k$.

Case 1: $d_j + d_k \leq n-2$.

Using (3), we obtain

$$d_j \leq (n-2) - d_k \leq (n-2) - (n-j-1) = j-1.$$

Take i, m so that $i = d_j = j - m$, where $m \geq 1$. By Case 1 we have $i \leq \frac{1}{2}(n-2)$. Since also $d_i = d_{j-m} \leq d_j = i$ and $d_{i+1} = d_{j-m+1} \leq d_j = i$, condition (iii) implies $d_{n-(j-m)-1} \geq n - (j-m) - 1$ or $d_{n-(j-m)} \geq n - (j-m)$. In either case,

$$d_{n-(j-m)} \geq n - (j-m) - 1. \tag{6}$$

Adding $d_j = j - m$ to (6), we obtain

$$d_j + d_{n-j+m} \geq n - 1. \tag{7}$$

But $d_j + d_k \leq n - 2$ and (7) together give $n - j + m > k$, hence $j + k < n + m$. On the other hand, (2) gives $j - m = d_j \geq n - k$, hence $j + k \geq n + m$, a contradiction. $\square$

CASE 2: $\quad d_j + d_k = n - 1$.

In this case we have equality in (5), hence all the inequalities in (4) become equalities. In particular, this implies that every cycle C_μ, $1 \leq \mu \leq t$, satisfies one of the following conditions:

(a) Every vertex in C_μ is adjacent to v_j (resp., v_k), and none are adjacent to v_k (resp., v_j), or

(b) $|V(C_\mu)|$ is even, and v_j, v_k are both adjacent to the same alternate vertices on C_μ.

We call a cycle of type (a) a *j-cycle* (resp., *k-cycle*), and a cycle of type (b) a *(j,k)-cycle*. Set $A = \bigcup_{j\text{-cycles } C} V(C)$, $B = \bigcup_{k\text{-cycles } C} V(C)$, and $D = \bigcup_{(j,k)\text{-cycles } C} V(C)$, and let $a \doteq |A|$, $b \doteq |B|$, and $c \doteq \frac{1}{2}|D|$.

Vertices in $V(G) - \{v_j, v_k\}$ which are adjacent to both (resp., neither) of v_j, v_k will be called *large* (resp., *small*) vertices. In particular, the vertices of each (j,k)-cycle are alternately large and small, and hence there are c small and c large vertices among the (j,k)-cycles.

By the definitions of a, b, c, noting that a cycle has at least 3 vertices, we have the following.

Observation 1. *We have $a = 0$ or $a \geq 3$, $b = 0$ or $b \geq 3$, and $c = 0$ or $c \geq 2$.*

By the choice of v_j, v_k and P, we also have the following observations.

Observation 2.

(a) If $(u, v_k) \notin E(G)$, then $d_G(u) \leq d_j$; if $(u, v_j) \notin E(G)$, then $d_G(u) \leq d_k$.

(b) A vertex in A has degree at most $d_j - 1$.

(c) A vertex in B has degree at most $d_k - 1$.

(d) A small vertex in D has degree at most $d_j - 1$.

Proof: Part (a) follows directly from the choice of v_j, v_k as nonadjacent with $d_G(v_j) + d_G(v_k) = d_j + d_k$ maximal.

For (b), consider any $a \in A$, with say $a \doteq v_\ell$. Since $(v_\ell, v_k) \notin E(G)$, we have $\ell < j$ by the maximality of $j+k$, and so $d_G(a) \leq d_j$. If $d_G(a) = d_j$, then since each vertex in A is adjacent to v_j, we can combine the path P and the j-cycle C_μ containing a (leaving the other cycles C_μ alone) into a path P' joining a and v_k such that $G - V(P')$ has a 2-factor and $|P'| > |P|$, contradicting the choice of P. Thus $d_G(a) \leq d_j - 1$, proving (b).

Parts (c) and (d) follow by a similar arguments. □

Let $p \doteq |V(P)|$, and let us re-index P as $v_j = w_1, w_2, \ldots, w_p = v_k$. By the case assumption, $d_P(w_1) + d_P(w_p) = p - 1$.

Assume first that $p = 3$. Then $d_j = a + c + 1$ and $d_k = b + c + 1$, so that $b \geq a$. Moreover, $n = a + b + 2c + 3$ and there are $c + 1$ large vertices and c small vertices.

If $b \geq 3$, the large vertex w_2 is not adjacent to a vertex in A or to a small vertex in D, or else G contains a 2-factor. Thus w_2 has degree at most $n - 1 - (a + c)$, and by Observations 2(b,c,d), π is majorized by

$$\pi_1 = (a+c)^{a+c}(a+c+1)^1(b+c)^b(b+c+1)^1(n-1-(a+c))^1(n-1)^c.$$

Setting $i = a + c$, so that $0 \leq i = a + c = (n-3) - (b+c) \leq \frac{1}{2}(n-3)$, π_1 becomes

$$\pi_1 = i^i(i+1)^1(n-i-3)^b(n-i-2)^1(n-i-1)^1(n-1)^c.$$

Since π_1 majorizes π, we have $d_i \leq i$, $d_{i+1} \leq i+1$, $d_{n-i-1} = d_{n-(a+c+1)} \leq n-i-2$, and $d_{n-i} = d_{n-(a+c)} \leq n-i-1$, and π violates condition (iii). Hence $b = 0$ by Observation 1, and a fortiori $a = 0$.

But if $a = b = 0$, then $c = \frac{1}{2}(n-3)$, n is odd, and by Observation 2(d), π is majorized by

$$\pi_2 = \left(\tfrac{1}{2}(n-3)\right)^{(n-3)/2}\left(\tfrac{1}{2}(n-1)\right)^2(n-1)^{(n-1)/2}.$$

Since π_2 majorizes π, we have $d_{(n+1)/2} \leq \frac{1}{2}(n-1)$, and π violates condition (i).

Hence we assume $p \geq 4$.

We make several further observations regarding the possible adjacencies of v_j, v_k into the path P.

Observation 3. *For all m, $1 \leq m \leq p-1$, we have $(w_1, w_{m+1}) \in E(G)$ if and only if $(w_p, w_m) \notin E(G)$.*

Proof: If $(w_1, w_{m+1}) \in E(G)$ then, $(w_p, w_m) \notin E(G)$, since otherwise $\langle V(P) \rangle$ is hamiltonian and G has a 2-factor. The converse follows since $d_P(w_1) + d_P(w_p) = p-1$. □

Observation 4. *If $(w_1, w_m), (w_1, w_{m+1}) \in E(G)$ for some m, $3 \leq m \leq p-3$, then we have $(w_1, w_{m+2}) \in E(G)$.*

Proof: If $(w_1, w_{m+2}) \notin E(G)$, then $(w_p, w_{m+1}) \in E(G)$ by Observation 3. But since $(w_1, w_m) \in E(G)$, this means that $\langle V(P) \rangle$ would have a 2-factor consisting of the cycles $(w_1, w_2, \ldots, w_m, w_1)$ and $(w_p, w_{m+1}, w_{m+2}, \ldots, w_p)$, and thus G would have a 2-factor, a contradiction. $\square$

Observation 4 implies that if w_1 is adjacent to consecutive vertices $w_m, w_{m+1} \in V(P)$ for some $m \geq 3$, then w_1 is adjacent to all of the vertices $w_m, w_{m+1}, \ldots, w_{p-1}$.

Observation 5. *If $(w_1, w_m), (w_1, w_{m-1}) \notin E(G)$ for some $5 \leq m \leq p-1$, then we have $(w_1, w_{m-2}) \notin E(G)$.*

Proof: If $(w_1, w_m) \notin E(G)$, then $(w_p, w_{m-1}) \in E(G)$ by Observation 3. So if also $(w_1, w_{m-2}) \in E(G)$, then $\langle V(P) \rangle$ would have a 2-factor as in the proof of Observation 4, leading to the same contradiction. $\square$

Observation 5 implies that if w_1 is not adjacent to two consecutive vertices w_{m-1}, w_m on P for some $m \leq p-1$, then w_1 is not adjacent to any of $w_3, \ldots, w_{m-1}, w_m$.

By Observation 3, the adjacencies of w_1 into P completely determine the adjacencies of w_p into P. But combining Observations 4 and 5, we see that the adjacencies of w_1 and w_p into P must appear as shown in Figure 1, for some $\ell, r \geq 0$. In summary, w_1 will be adjacent to $r \geq 0$ consecutive vertices $w_{p-r}, \ldots, w_{p-1}$ (where $w_\alpha, \ldots, w_\beta$ is taken to be empty if $\alpha > \beta$), w_p will be adjacent to $\ell \geq 0$ consecutive vertices $w_2, \ldots, w_{\ell+1}$, and w_1, w_p are each adjacent to the vertices $w_{\ell+3}, w_{\ell+5}, \ldots, w_{p-r-4}, w_{p-r-2}$. Note that $\ell = p-2$ implies $r = 0$, and $r = p-2$ implies $\ell = 0$.

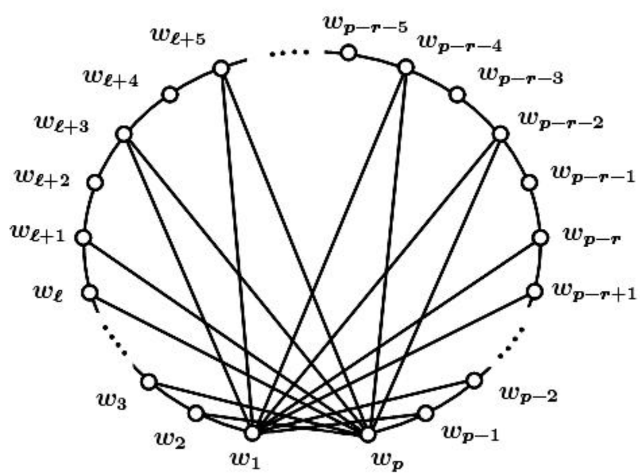

Figure 1: The adjacencies of w_1, w_p on P.

Counting neighbors of w_1 and w_p we get their degrees as follows.

Observation 6.

$$d_j = d_G(w_1) = \begin{cases} a+c+1, & \textit{if } \ell = p-2,\ r=0, \\ a+c+p-2, & \textit{if } r = p-2,\ \ell = 0, \\ a+c+r+\frac{1}{2}(p-r-\ell-1); & \textit{otherwise}; \end{cases}$$

$$d_k = d_G(w_p) = \begin{cases} b+c+p-2, & \textit{if } \ell = p-2,\ r=0, \\ b+c+1, & \textit{if } r = p-2,\ \ell = 0, \\ b+c+\ell+\frac{1}{2}(p-r-\ell-1); & \textit{otherwise}. \end{cases}$$

We next prove some observations to limit the possibilities for (a,b) and (ℓ,r).

Observation 7. *If $(w_1,w_{p-1}) \in E(G)$ (resp., $(w_2,w_p) \in E(G)$), then we have $b=0$ (resp., $a=0$).*

Proof: If $b \neq 0$, there exists a k-cycle $C \doteq (x_1,x_2,\ldots,x_s,x_1)$. But if also $(w_1,w_{p-1}) \in E(G)$, then $(w_1,w_2,\ldots,w_{p-1},w_1)$ and $(w_p,x_1,\ldots,x_s,w_p)$ would be a 2-factor in $\langle V(C) \cup V(P)\rangle$, implying a 2-factor in G. The proof that $(w_2,w_p) \in E(G)$ implies $a=0$ is symmetric. $\square$

From Observation 6, we have

$$0 \le d_k - d_j = b - a + \begin{cases} p-3, & \text{if } \ell = p-2,\ r=0, \\ 3-p, & \text{if } r = p-2,\ \ell=0, \\ \ell - r, & \text{otherwise.} \end{cases} \tag{8}$$

From this, we obtain

Observation 8. *$\ell \ge r$.*

Proof: Suppose first $r \neq p-2$. If $r > \ell \ge 0$, then $b > a \ge 0$ since $b + \ell \ge a + r$ by (8). But $r > 0$ implies $(w_1,w_{p-1}) \in E(G)$, and thus $b=0$ by Observation 7, a contradiction.

Suppose then $r = p-2 \ge 2$. Then $b > a \ge 0$, since $b \ge a+p-3$ by (8). Since $r > 0$, we have the same contradiction as in the previous paragraph. $\square$

Observation 9. *If $r \ge 1$, then $\ell \le 1$.*

Proof: Else we have $(w_1,w_{p-1}),(w_p,w_2),(w_p,w_3) \in E(G)$, and $(w_1,w_2,w_p,w_3,\ldots,w_{p-1},w_1)$ would be a hamiltonian cycle in $\langle V(P)\rangle$. Thus G would have a 2-factor, a contradiction. $\square$

Observations 8 and 9 together limit the possibilities for (ℓ,r) to $(1,1)$ and $(\ell,0)$ with $0 \le \ell \le p-2$. We also cannot have $(\ell,r) = (p-3,0)$, since w_p is always adjacent to w_{p-1}, and so we would have $\ell = p-2$ in that case. And we cannot have $(\ell, r) = (p-4, 0)$, since then $p - r - \ell - 1$ is odd, violating Observation 6. To complete the proof of Theorem 3.1, we will deal with the remaining possibilities in a number of cases, and show that all of them lead to a contradiction of one or more of conditions (i) through (iv).

Before doing so, let us define the spanning subgraph H of G by letting $E(H)$ consist of the edges in the cycles C_μ, $0 \leq \mu \leq t$, or in the path P, together with the edges incident to w_1 or w_p. Note that the edges incident to w_1 or w_p completely determine the large or small vertices in G. In the proofs of the cases below, any adjacency beyond those indicated would create an edge e such that $H + e$, and a fortiori G, contains a 2-factor.

Case 2.1: $(\ell, r) = (1, 1)$.

Since $(w_1, w_{p-1}), (w_2, w_p) \in E(G)$, we have $a = b = 0$, by Observation 7. Using Observation 6 this means that $d_j = d_k = \frac{1}{2}(n-1)$, and hence n is odd. Additionally, there are $c + \frac{1}{2}(p-3) = \frac{1}{2}(n-3)$ small vertices. Each of these small vertices has degree at most d_j by Observation 2 (a), and so π is majorized by

$$\pi_3 = \left(\tfrac{1}{2}(n-1)\right)^{(n+1)/2}(n-1)^{(n-1)/2}.$$

But π_3 (a fortiori π) violates condition (i). □

Case 2.2: $(\ell, r) = (0, 0)$.

By Observation 6, $d_j = a + c + \frac{1}{2}(p-1)$ and $d_k = b + c + \frac{1}{2}(p-1)$, so that $b \geq a$. Also, there are $c + \frac{1}{2}(p-3)$ large and $c + \frac{1}{2}(p-5)$ small vertices.

- By Observation 2 (b,c), each vertex in A (resp., B) has degree at most $d_j - 1 = a + c + \frac{1}{2}(p-3)$ (resp., $d_k - 1 = b + c + \frac{1}{2}(p-3)$).
- Each small vertex is adjacent to at most the large vertices (otherwise G contains a 2-factor), and so each small vertex has degree at most $c + \frac{1}{2}(p-3)$.
- The vertex w_2 (resp., w_{p-1}) is adjacent to at most the large vertices and w_1 (resp., w_p) (otherwise G contains a 2-factor), and so w_2, w_{p-1} each have degree at most $c + \frac{1}{2}(p-1)$.

Thus π is majorized by

$$\begin{aligned}\pi_4 = {} & \left(c + \tfrac{1}{2}(p-3)\right)^{c+(p-5)/2}\left(c + \tfrac{1}{2}(p-1)\right)^2\left(a + c + \tfrac{1}{2}(p-3)\right)^a \\ & \left(a + c + \tfrac{1}{2}(p-1)\right)^1\left(b + c + \tfrac{1}{2}(p-3)\right)^b\left(b + c + \tfrac{1}{2}(p-1)\right)^1(n-1)^{c+(p-3)/2}.\end{aligned}$$

Setting $i = a + c + \frac{1}{2}(p-1)$, so that $2 \leq i = \frac{1}{2}(n - (b-a) - 1) \leq \frac{1}{2}(n-1)$, the sequence π_4 becomes

$$\pi_4 = (i-a-1)^{i-a-2}(i-a)^2(i-1)^a i^1(n-i-2)^{n-2i+a-1}(n-i-1)^1(n-1)^{i-a-1}.$$

If $2 \leq i \leq \frac{1}{2}(n-2)$, then since π_4 majorizes π, we have $d_i \leq i$, $d_{i+1} \leq i$, $d_{n-i-1} \leq n-i-2$, and $d_{n-i} \leq n-i-2$, and π violates condition (iii).

If $i = \frac{1}{2}(n-1)$, then n is odd, and π_4 reduces to

$$\pi_4' = \left(\tfrac{1}{2}(n-3)-a\right)^{(n-5)/2-a}\left(\tfrac{1}{2}(n-1)-a\right)^2\left(\tfrac{1}{2}(n-3)\right)^{2a}\left(\tfrac{1}{2}(n-1)\right)^2(n-1)^{(n-3)/2-a}.$$

Since π_4' majorizes π, we have $d_{(n+1)/2} \le \frac{1}{2}(n-1)$, and π violates condition (i). $\square$

CASE 2.3: $(\ell, r) = (1, 0)$

By Observation 7, $a = 0$, and thus by Observation 6, $d_j = c + \frac{1}{2}(p-2)$ and $d_k = b + c + \frac{1}{2}p$. Also, there are $c + \frac{1}{2}(p-2)$ large and $c + \frac{1}{2}(p-4)$ small vertices. If $p = 4$ then $\ell = 2$, a contradiction, and hence $p \ge 6$.

- By Observation 2 (c), each vertex in B has degree at most $d_k - 1 = b + c + \frac{1}{2}(p-2)$.
- Each small vertex is adjacent to at most the large vertices, and so each small vertex has degree at most $c + \frac{1}{2}(p-2)$.
- The vertex w_{p-1} is adjacent to at most w_p and the large vertices, and so w_{p-1} has degree at most $c + \frac{1}{2}p$.

Thus π is majorized by

$$\pi_5 = \left(c + \tfrac{1}{2}(p-2)\right)^{c+(p-2)/2}\left(c + \tfrac{1}{2}p\right)^1\left(b + c + \tfrac{1}{2}(p-2)\right)^b\left(b + c + \tfrac{1}{2}p\right)^1(n-1)^{c+(p-2)/2}.$$

Setting $i = c + \frac{1}{2}(p-2)$, so that $2 \le i = \frac{1}{2}(n-b-2) \le \frac{1}{2}(n-2)$, π_5 becomes

$$\pi_5 = i^i(i+1)^1(n-i-2)^{n-2i-2}(n-i-1)^1(n-1)^i.$$

If $2 \le i \le \frac{1}{2}(n-3)$, then since π_5 majorizes π, we have $d_i \le i$, $d_{i+1} \le i+1$, $d_{n-i-1} \le n-i-2$, and $d_{n-i} \le n-i-1$, and π violates condition (iii).
If $i = \frac{1}{2}(n-2)$, then n is even, and π_5 reduces to

$$\pi_5' = \left(\tfrac{1}{2}n - 1\right)^{n/2-1}\left(\tfrac{1}{2}n\right)^2(n-1)^{n/2-1}.$$

Since π_5' majorizes π, we have $d_{n/2-1} \le \frac{1}{2}n - 1$ and $d_{n/2+1} \le \frac{1}{2}n$, and π violates condition (ii). $\square$

CASE 2.4: $(\ell, r) = (\ell, 0)$, *where* $2 \le \ell \le p-5$

We have $a = 0$ by Observation 7, and $p - \ell \ge 5$ by Case 2.4. By Observation 6, $d_j = c + \frac{1}{2}(p-\ell-1)$ and $d_k = b + c + \ell + \frac{1}{2}(p-\ell-1)$. Moreover, there are $c + \frac{1}{2}(p-\ell-1)$ large vertices including w_2, and $c + \frac{1}{2}(p-\ell-3)$ small vertices.

- By Observation 2 (c), each vertex in B has degree at most $d_k - 1 = b + c + \ell + \frac{1}{2}(p-\ell-3)$.
- Each small vertex other than $w_{\ell+2}$ is adjacent to at most the large vertices except w_2, and so each small vertex other than $w_{\ell+2}$ has degree at most $c + \frac{1}{2}(p-\ell-3)$.
- The vertex $w_{\ell+2}$ is not adjacent to w_p, and so by Observation 2 (a), $w_{\ell+2}$ has degree at most $d_j = c + \frac{1}{2}(p-\ell-1)$.

- The vertex w_{p-1} is adjacent to at most w_p and the large vertices except w_2, and so w_{p-1} has degree at most $c + \frac{1}{2}(p-\ell-1)$.
- Each w_m, $3 \le m \le \ell$, is adjacent to at most w_p, the large vertices, the vertices in B, and $\{w_3, \dots, w_{\ell+1}\} - \{w_m\}$. Hence each such w_m has degree at most $b + c + \ell + \frac{1}{2}(p-\ell-3)$.
- The vertex w_2 is adjacent to at most w_1, w_p, the other large vertices, the vertices in B, and $\{w_3, \dots, w_{\ell+1}\}$. Hence w_2 has degree at most $b + c + \ell + \frac{1}{2}(p-\ell-1)$.
- The vertex $w_{\ell+1}$ is not adjacent to w_1, and so by Observation 2 (a), vertex $w_{\ell+1}$ has degree at most $d_k = b + c + \ell + \frac{1}{2}(p-\ell-1)$.

Thus π is majorized by

$$\pi_6 = \left(c + \tfrac{1}{2}(p-\ell-3)\right)^{c+(p-\ell-5)/2} \left(c + \tfrac{1}{2}(p-\ell-1)\right)^3$$
$$\left(b + c + \ell + \tfrac{1}{2}(p-\ell-3)\right)^{b+\ell-2} \left(b + c + \ell + \tfrac{1}{2}(p-\ell-1)\right)^3 (n-1)^{c+(p-\ell-3)/2}.$$

Setting $i = c - 1 + \frac{1}{2}(p-\ell-1)$, so that $1 \le i = \frac{1}{2}(n-b-\ell-3) \le \frac{1}{2}(n-5)$, π_6 becomes

$$\pi_6 = i^{i-1}(i+1)^3(i+b+\ell)^{b+\ell-2}(i+b+\ell+1)^3(n-1)^i.$$

Since π_6 majorizes π, we have $d_{i-1} \le i$, $d_{i+2} \le i+1$, $d_{n-i-3} \le i+b+\ell = n-i-3$, and $d_{n-i} \le i+b+\ell+1 = n-i-2$, and thus π violates condition (iv). □

Case 2.5: $(\ell, r) = (p-2, 0)$

We have $a = 0$, by Observation 7. By Observation 6, we then have $d_j = c+1$ and $d_k = b+c+p-2$. If $d_1 \le 1$, then condition (iii) with $i = 0$ implies $d_{n-1} \ge n-1$, which means there are at least 2 vertices adjacent to all other vertices, a contradiction. Hence $c + 1 = d_j \ge d_1 \ge 2$, and so $c \ge 2$ by Observation 1. Finally, there are $c+1$ large vertices including w_2, and c small vertices.

- By Observation 2 (a), the vertices in B have degree at most $d_k = b+c+p-2$.
- By Observation 2 (d), the small vertices in D have degree at most $d_j - 1 = c$.
- The vertex w_2 is not adjacent to the small vertices in D, and so w_2 has degree at most $n-1-c = b+c+p-1$.
- The vertices $w_3, \dots, w_{p-1}$ have degree at most $d_k = b+c+p-2$ by Observation 2 (a), since none of them are adjacent to $w_1 = v_j$.

Thus π is majorized by

$$\pi_7 = c^c(c+1)^1(b+c+p-2)^{b+p-2}(b+c+p-1)^1(n-1)^c.$$

Setting $i = c$, so that $2 \le c = i = \frac{1}{2}(n-b-p) \le \frac{1}{2}(n-4)$, π_7 becomes

$$\pi_7 = i^i(i+1)^1(n-i-2)^{n-2i-2}(n-i-1)^1(n-1)^i.$$

Since π_7 majorizes π, we have $d_i \le i$, $d_{i+1} \le i+1$, $d_{n-i-1} \le n-i-2$, and $d_{n-i} \le n-i-1$, and π violates condition (iii). □

The proof of Theorem 3.1 is complete. ■

4 Sufficient condition for the existence of a k-factor, $k \geq 2$

The increase in complexity of Theorem 3.1 ($k = 2$) compared to Corollay 2.2 ($k = 1$) suggests that the best monotone condition for π to be forcibly k-factor graphical may become unwieldy as k increases. Indeed, we make the following conjecture.

Conjecture 4.1. *The best monotone condition for a degree sequence π of length n to be forcibly k-factor graphical requires checking at least $f(k)$ nonredundant conditions (where each condition may require $O(n)$ checks), where $f(k)$ grows superpolynomially in k.*

Kriesell [10] has verified such rapidly increasing complexity for the best monotone condition for π to be forcibly k-edge-connected. Indeed, Kriesell has shown such a condition entails checking at least $p(k)$ nonredundant conditions, where $p(k)$ denotes the number of partitions of k. It is well-known [8] that $p(k) \sim \dfrac{e^{\pi\sqrt{2k/3}}}{4\sqrt{3}k}$.

The above conjecture suggests the desirability of obtaining a monotone condition for π to be forcibly k-factor graphical which does not require checking a superpolynomial number of conditions. Our goal in this section is to prove such a condition for $k \geq 2$. Since our condition will require Tutte's Factor Theorem [2, 16], we begin with some needed background.

Belck [2] and Tutte [16] characterized graphs G that do not contain a k-factor. For disjoint subsets A, B of $V(G)$, let $C = V(G) - A - B$. We call a component H of $\langle C \rangle$ *odd* if $k|H| + e(H, B)$ is odd. The number of odd components of $\langle C \rangle$ is denoted by $odd_k(A, B)$. Define

$$\Theta_k(A, B) \doteq k|A| + \sum_{u \in B} d_{G-A}(u) - k|B| - odd_k(A, B).$$

Theorem 4.2. *Let G be a graph on n vertices and $k \geq 1$.*

(a) [16]. *For any disjoint $A, B \subseteq V(G)$, $\Theta_k(A, B) \equiv kn \pmod 2$;*

(b) [2, 16]. *The graph G does not contain a k-factor if and only if $\Theta_k(A, B) < 0$, for some disjoint $A, B \subseteq V(G)$.*

We call any disjoint pair $A, B \subseteq V(G)$ for which $\Theta_k(A, B) < 0$ a *k-Tutte-pair* for G. Note that if kn is even, then A, B is a k-Tutte-pair for G if and only if

$$k|A| + \sum_{u \in B} d_{G-A}(u) \leq k|B| + odd_k(A, B) - 2.$$

Moreover, for all $u \in B$ we have $d_G(u) \leq d_{G-A}(u) + |A|$, so $\sum_{u \in B} d_G(u) \leq \sum_{u \in B} d_{G-A}(u) + |A||B|$. Thus for each k-Tutte-pair A, B we have

$$\sum_{u \in B} d_G(u) \leq k|B| + |A||B| - k|A| + odd_k(A, B) - 2. \tag{9}$$

Our main result in this section is the following condition for a graphical degree sequence π to be forcibly k-factor graphical. The condition will guarantee that no k-Tutte-pair can exist, and is readily seen to be monotone. We again set $d_0 = 0$.

Theorem 4.3. *Let $\pi = (d_1 \le \cdots \le d_n)$ be a graphical degree sequence, and let $k \ge 2$ be an integer such that kn is even. Suppose*

(i) $d_1 \ge k$;

(ii) for all a, b, q with $0 \le a < \frac{1}{2}n$, $0 \le b \le n-a$ and $\max\{0, a(k-b)+2\} \le q \le n-a-b$ so that $\sum_{i=1}^{b} d_i \le kb + ab - ka + q - 2$, the following holds: Setting $r = a + k + q - 2$ and $s = n - \max\{0, b-k+1\} - \max\{0, q-1\} - 1$, we have

$$(*) \quad r \le s \text{ and } d_b \le r, \text{ or } r > s \text{ and } d_{n-a-b} \le s \implies d_{n-a} \ge \max\{r, s\} + 1.$$

Then π is forcibly k-factor graphical.

Proof: Let n and $k \ge 2$ be integers with kn even. Suppose π satisfies (i) and (ii) in the theorem, but has a realization G with no k-factor. This means that G has at least one k-Tutte-pair.

Following [7], a k-Tutte-pair A, B is *minimal* if either $B = \varnothing$, or $\Theta_k(A, B') \ge 0$ for all proper subsets $B' \subset B$. We then have

Lemma 4.4 ([7])**.** *Let $k \ge 2$, and let A, B be a minimal k-Tutte-pair for a graph G with no k-factor. If $B \ne \varnothing$, then $\Delta(\langle B \rangle) \le k-2$.*

Next let A, B be a k-Tutte-pair for G with A as large as possible, and A, B minimal. Also, set $C = V(G) - A - B$. We establish some further observations.

Lemma 4.5.

(a) $|A| < \frac{1}{2}n$.

(b) For all $v \in C$, $e(v, B) \le \min\{k-1, |B|\}$.

(c) For all $u \in B$, $d_G(u) \le |A| + k + odd_k(A, B) - 2$.

Proof: Suppose $|A| \ge \frac{1}{2}n$, so that $|A| \ge |B| + |C|$. Then we have

$$\begin{aligned} \Theta_k(A, B) &= k|A| + \sum_{u \in B} d_{G-A}(u) - k|B| - odd_k(A, B) \ge k(|A| - |B|) - odd_k(A, B) \\ &\ge k|C| - odd_k(A, B) > |C| - odd_k(A, B) \ge 0, \end{aligned}$$

which contradicts that A, B is a k-Tutte-pair.

For (b), clearly $e(v, B) \le |B|$. If $e(v, B) \ge k$ for some $v \in C$, move v to A, and consider the change in each term in $\Theta_k(A, B)$:

$$\underbrace{k|A|}_{\text{increases by } k} + \underbrace{\sum_{u \in B} d_{G-A}(u)}_{\text{decreases by } e(v,B) \ge k} - k|B| - \underbrace{odd_k(A, B)}_{\text{decreases by } \le 1}.$$

So by Theorem 4.2 (a), $A \cup \{v\}, B$ is also a k-Tutte-pair in G, contradicting the assumption that A, B is a k-Tutte-pair with A as large as possible.

And for (c), suppose that $d_G(t) \geq |A| + k + odd_k(A, B) - 1$ for some $t \in B$. This implies that $d_{G-A}(t) \geq k + odd_k(A, B) - 1$. Now move t to C, and consider the change in each term in $\Theta_k(A, B)$:

$$k|A| + \underbrace{\sum_{u \in B} d_{G-A}(u)}_{\substack{\text{decreases by} \\ d_{G-A}(t) \geq k + odd_k(A,B) - 1}} - \underbrace{k|B|}_{\text{decreases by } k} - \underbrace{odd_k(A, B)}_{\text{decreases by } \leq odd_k(A,B)} .$$

So by Theorem 4.2 (a), $A, B - \{t\}$ is also a k-Tutte-pair for G, contradicting the minimality of A, B. $\square$

We introduce some further notation. Set $a \doteq |A|$, $b \doteq |B|$, $c \doteq |C| = n - a - b$, $q \doteq odd_k(A, B)$, $r \doteq a + k + q - 2$, and $s \doteq n - \max\{0, b - k + 1\} - \max\{0, q - 1\} - 1$. Using this notation, (9) can be written as

$$\sum_{u \in B} d_G(u) \leq kb + ab - ka + q - 2. \tag{10}$$

By Lemma 4.5 (a) we have $0 \leq a < \frac{1}{2}n$. Since B is disjoint from A, we trivially have $0 \leq b \leq n - a$. And since the number of odd components of C is at most the number of elements of C, we are also guaranteed that $q \leq n - a - b$. Finally, since for all vertices v we have $d_G(v) \geq d_1 \geq k$, we get from (10) that $q \geq \sum_{u \in B} d_G(u) - kb - ab + ka + 2 \geq kb - kb - ab + ka + 2 = a(k - b) + 2$, hence $q \geq \max\{0, a(k - b) + 2\}$. It follows that a, b, q satisfy the conditions in Theorem 4.3 (ii).

Next, by Lemma 4.5 (c) we have that

$$\text{for all } u \in B: \quad d_G(u) \leq r. \tag{11}$$

If $C \neq \varnothing$ (i.e., if $a + b < n$), let m be the size of a largest component of $\langle C \rangle$. Then, using Lemma 4.5 (b), for all $v \in C$ we have

$$\begin{aligned} d_G(v) &= e(v, A) + e(v, B) + e(v, C) \leq |A| + \min\{k - 1, |B|\} + m - 1 \\ &= a + b - \max\{0, b - k + 1\} + m - 1. \end{aligned}$$

Clearly $m \leq |C| = n - a - b$. If $q \geq 1$, then $m \leq n - a - b - (q - 1)$, since C has at least q components. Thus $m \leq n - a - b - \max\{0, q - 1\}$. Combining this all gives

$$\text{for all } v \in C: \quad d_G(v) \leq n - \max\{0, b - k + 1\} - \max\{0, q - 1\} - 1 = s. \tag{12}$$

Next notice that we cannot have $n - a = 0$, because otherwise $B = C = \varnothing$ and $odd_k(A, B) = 0$, and (9) becomes $0 \leq -ka - 2$, a contradiction. From (11) and (12) we see that each of the $n - a > 0$ vertices in $B \cup C$ has degree at most $\max\{r, s\}$, and so $d_{n-a} \leq \max\{r, s\}$.

If $r \le s$, then each of the b vertices in B has degree at most r, and so $d_b \le r$. This also holds if $b = 0$, since we set $d_0 = 0$, and $r = a + k + q - 2 \ge 0$ because $k \ge 2$.

If $r > s$, then each of $n - a - b$ vertices in C has degree at most s by (12), and so $d_{n-a-b} \le s$. This also holds if $n - a - b = 0$, since we set $d_0 = 0$ and

$$\begin{aligned} s &= n - \max\{0, b - k + 1\} - \max\{0, q - 1\} - 1 \\ &\ge \min\{n - 1, n - q, (n - b) + (k - 2), (n - q - b) + (k - 1)\} \ge 0, \end{aligned}$$

since $k \ge 2$ and $q \le n - a - b$.

So we always have $r \le s$ and $d_b \le r$, or $r > s$ and $d_{n-a-b} \le s$, but also $d_{n-a} \le \max\{r, s\}$, contradicting assumption (ii) $(*)$ in Theorem 4.3. ■

How good is Theorem 4.3? We know it is not best monotone for $k = 2$. For example, the sequence $\pi = 4^4 6^3 10^4$ satisfies Theorem 3.1, but not Theorem 4.3 (it violates $(*)$ when $a = 4$, $b = 5$ and $q = 2$, with $r = 6$ and $s = 5$). And it is very unlikely the theorem is best monotone for any $k \ge 3$. Nevertheless, Theorem 4.3 appears to be quite tight. In particular, we conjecture for each $k \ge 2$ there exists a $\pi = (d_1 \le \cdots \le d_n)$ such that

- (π, k) satisfies Theorem 4.3, and
- there exists a degree sequence π', with $\pi' \le \pi$ and $\sum_{i=1}^{n} d_i' = \left(\sum_{i=1}^{n} d_i\right) - 2$, such that π' is not forcibly k-factor graphical.

Informally, for each $k \ge 2$, there exists a pair (π, π') with π' 'just below' π such that Theorem 4.3 detects that π is forcibly k-factor graphical, while π' is not forcibly k-factor graphical.

For example, let $n \equiv 2 \pmod 4$ and $n \ge 6$, and consider the sequences $\pi_n \doteq \left(\frac{1}{2}n\right)^{n/2+1}(n-1)^{n/2-1}$ and $\pi_n' \doteq \left(\frac{1}{2}n - 1\right)^2\left(\frac{1}{2}n\right)^{n/2-1}(n-1)^{n/2-1}$. It is easy to verify that the unique realization of π_n' fails to have a k-factor, for $k = \frac{1}{4}(n+2) \ge 2$. On the other hand, we have programmed Theorem 4.3, and verified that π_n satisfies Theorem 4.3 with $k = \frac{1}{4}(n+2)$ for all values of n up to $n = 2502$. We conjecture that $(\pi_n, \frac{1}{4}(n+2))$ satisfies Theorem 4.3 for all $n \ge 6$ with $n \equiv 2 \pmod 4$.

There is another sense in which Theorem 4.3 seems quite good. A graph G is *t-tough* if $t \cdot \omega(G) \le |X|$, for every $X \subseteq V(G)$ with $\omega(G - X) > 1$, where $\omega(G - X)$ denotes the number of components of $G - X$. In [1], the authors give the following best monotone condition for π to be forcibly t-tough, for $t \ge 1$.

Theorem 4.6 ([1]). *Let $t \ge 1$, and let $\pi = (d_1 \le \cdots \le d_n)$ be graphical with $n > (t+1)\lceil t \rceil / t$. If*

$$d_{\lfloor i/t \rfloor} \le i \implies d_{n-i} \ge n - \lfloor i/t \rfloor, \qquad \textit{for } t \le i < tn/(t+1),$$

then π is forcibly t-tough graphical.

We also have the following classical result.

Theorem 4.7 ([7]). *Let $k \geq 1$, and let G be a graph on $n \geq k+1$ vertices with kn even. If G is k-tough, then G has a k-factor.*

Based on checking many examples with our program, we conjecture that there is a relation between Theorems 4.6 and 4.3, which somewhat mirrors Theorem 4.7.

Conjecture 4.8. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be graphical, and let $k \geq 2$ be an integer with $n > k+1$ and kn even. If π is forcibly k-tough graphical by Theorem 4.6, then π is forcibly k-factor graphical by Theorem 4.3.*

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