Graphs with the maximum or minimum number of 1-factors

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Abstract

Recently Alon and Friedland have shown that graphs which are the union of complete regular bipartite graphs have the maximum number of 1-factors over all graphs with the same degree sequence. We identify two families of graphs that have the maximum number of 1-factors over all graphs with the same number of vertices and edges: the almost regular graphs which are unions of complete regular bipartite graphs, and complete graphs with a matching removed. The first family is determined using Alon and Friedland's bound. For the second family, we show that a graph transformation which is known to increase network reliability also increases the number of 1-factors. In fact, more is true: this graph transformation increases the number of k-factors for all $k \geq 1$, and "in reverse" also shows that in general, threshold graphs have the fewest k-factors. We are then able to determine precisely which threshold graphs have the fewest 1-factors. We conjecture that the same graphs have the fewest k-factors for all $k \geq 2$ as well.

1 Introduction

For any terms or notation not defined in this paper, refer to [14]. We consider simple graphs $G = (V, E)$ only, where $V(G)$ and $E(G)$ denote the vertex and edge sets of G respectively, and we set $|V(G)| = n(G)$ and $|E(G)| = m(G)$. The set of all vertices adjacent to a vertex $v \in V(G)$, or *neighbors* of $v \in V(G)$, is called the *neighborhood* of $v \in V(G)$ and is denoted by $N_G(v)$. The number $|N_G(v)|$ is called the *degree* of the vertex v of G, and is denoted by $d_G(v)$. For brevity, when the choice of G is clear, we abbreviate $n(G)$, $m(G)$, $N_G(v)$, and $d_G(v)$ as n, m, $N(v)$, and $d(v)$, respectively. The *degree sequence* of G is the n-tuple $(d_1, \ldots, d_n)$ consisting of the degrees of the vertices of G written in non-decreasing order. We call a graph G *regular* if $d_1 = d_n$, and *almost regular* if $d_n - d_1 \leq 1$. Note that, under this terminology, regular graphs are a subset of almost regular graphs. A *k-factor* of a graph G is a spanning subgraph H of G with $d_H(v) = k$ for all $v \in V(G)$. A *matching* of G is a set of independent edges of G. A 1-factor of G is also called a *perfect matching* of G. For any fixed $k \geq 1$, the number of k-factors of G is denoted $\Phi_k(G)$, although it is conventional to write $\Phi_1(G) = \Phi(G)$. A standard reference on matchings is [6].

Recently Alon and Friedland [1] gave an upper bound on the number of 1-factors of a graph G, based on the degree sequence of G. If we partition the set of all graphs into equivalence classes according to their degree sequences, any graph achieving the Alon and Friedland bound necessarily has the maximum number of 1-factors in its equivalence class. Alon and Friedland also showed that the bound is achieved if and only if G is the union of complete regular bipartite graphs, and so from this view those graphs have the maximum number of 1-factors.

Identifying graphs which have the maximum number of a particular subgraph, or type of subgraph, is a problem that has a long history in graph theory and its applications; finding graphs with the maximum number of spanning trees, for instance, is a difficult subproblem in all-terminal network reliability problem, see e.g., [10]. (The all-terminal network reliability problem seeks a graph topology G that maximizes the probability that G is connected, given that the edges of G may fail.) However in network reliability applications—and indeed, as with many problems of this type—the set of all graphs is typically partitioned into equivalence classes based on the number of vertices and edges, rather than degree sequence. In this paper we take this view and identify two families of graphs which have the maximum number of 1-factors in $G_{n,m}$, the class of graphs with n vertices and m edges.

The paper is organized as follows. In the next section, we show that almost regular graphs which are the union of complete regular bipartite graphs have the maximum number of 1-factors of any graph in their class. This is done by maximizing the bound function given in [1] for degree sequences with fixed sums. In Section 3 we show that for any $k \geq 1$, a complete graph with a matching removed has the maximum number of k-factors of any graph in the class $G_{n,m}$, $m \geq \binom{n}{2} - n/2$. This is accomplished by showing that a graph transformation which is known to increase network reliability also increases the number of k-factors. In the fourth section we use this graph transformation "in reverse" to show that, for any n, m, there is a threshold graph $G \in G_{n,m}$ which has the minimum number of k-factors in its class. For $k = 1$, we identify the particular threshold graphs with the minimum number of 1-factors. We conjecture that the same graphs have the minimum number of k-factors for any $k \geq 2$ as well.

2 Graphs with the maximum number of 1-factors

We begin with the previously mentioned Alon and Friedland result.

Theorem 2.1 *[1] For any graph G with degree sequence $\pi = (d_1, \ldots, d_n)$, we have*

$$\Phi(G) \leq \left(\prod_{i=1}^{n} (d_i!)^{\frac{1}{d_i}} \right)^{\frac{1}{2}}$$

where we take $(0!)^{\frac{1}{0}} = 0$. The bound is achieved if and only if G is the union of complete regular bipartite graphs.

Determining which graphs in $G_{n,m}$ have the maximum number of 1-factors involves, of course, determining which degree sequences with fixed sum $2m$ maximize the bound function $\prod_{i=1}^{n} (d_i!)^{\frac{1}{d_i}}$. This is done in the next lemma.

Lemma 2.2 *For fixed n, m, let $D_{n,m} = \{\pi = (d_1, \ldots, d_n) \,|\, \pi$ is a degree sequence of $G \in G_{n,m}\}$. The function $f : D_{n,m} \to \mathbb{R}$ defined by*

$$f(\pi) = \prod_{i=1}^{n} (d_i!)^{\frac{1}{d_i}}$$

is maximized when π is the degree sequence of an almost regular graph.

Proof: If π is not the degree sequence of a regular or almost regular graph G, then for some indices i, j we have $d_i > d_j + 1$. To prove the lemma we show that, if this occurs, we may replace d_i, d_j with $d_i' = d_i - 1$ and $d_j' = d_j + 1$, and for the resulting degree sequence π' we will have $f(\pi') > f(\pi)$.

Let $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ denote Euler's gamma function, and recall that $\Gamma(x+1) = x!$ for positive integers x. A continuous function $g(x)$ is *strictly concave* on an interval I if $g''(x) < 0$ for all $x \in I$. In [3] it was shown that the function

$$g(x) = \frac{\log \Gamma(x+1)}{x}$$

was continuous and strictly concave on $(-1, \infty)$. As a consequence, $g'(x)$ is a decreasing function on $(-1, \infty)$, and therefore the function

$$h(x) = \int_{x-1}^{x} g'(t)\, dt = \frac{\log \Gamma(x+1)}{x} - \frac{\log \Gamma(x)}{x-1}$$

is decreasing as well. Thus for positive integers $x > y$ we have $h(y) - h(x) > 0$, or

$$\frac{\log y!}{y} - \frac{\log (y-1)!}{y-1} - \left(\frac{\log x!}{x} - \frac{\log (x-1)!}{x-1} \right) > 0.$$

After some algebraic manipulation, this reveals that for positive integers $x > y$,

$$((x-1)!)^{\frac{1}{x-1}} (y!)^{\frac{1}{y}} > (x!)^{\frac{1}{x}} ((y-1)!)^{\frac{1}{y-1}}.$$

Letting $x = d_i$ and $y = d_j + 1$ completes the proof. □

Before we state our final result of this section, we note that for any fixed n, m, there is a unique degree sequence $(d_1, \ldots, d_n)$ that is almost regular such that $2m = \sum_{i=1}^{n} d_i$. (This was shown, for instance, in [11], again in the context of spanning trees.) Thus among the degree sequences corresponding to the graphs in any $G_{n,m}$ class, there is exactly one almost regular degree sequence.

Theorem 2.3 *Almost regular graphs which are the union of complete regular bipartite graphs have the maximum number of 1-factors in their class.*

Proof: Let n, m be positive integers such that there exists an almost regular $G \in G_{n,m}$ which is the union of complete regular bipartite graphs, let $\pi = (d_1, \ldots, d_n)$ denote the degree sequence of that G, and note that π is the unique almost regular degree sequence corresponding to graphs in this class. Let $H \in G_{n,m}$, $H \neq G$, and let $\pi' = (d_1', \ldots, d_n')$ be the degree sequence of

H. Assume to the contrary that $\Phi(G) < \Phi(H)$. Now either $\pi = \pi'$ or $\pi \neq \pi'$. If $\pi = \pi'$, then by Theorem 2.1, we have $\Phi(G) = \Phi(H)$ if and only if $G = H$, contradicting $G \neq H$. If $\pi \neq \pi'$, then by the uniqueness of π, we have π' is not almost regular. By Theorem 2.1 and Lemma 2.2,

$$\Phi(G) = \left(\prod_{i=1}^{n}(d_i!)^{\frac{1}{d_i}}\right)^{\frac{1}{2}} > \left(\prod_{i=1}^{n}(d_i'!)^{\frac{1}{d_i'}}\right)^{\frac{1}{2}} = \Phi(H),$$

contradicting $\Phi(G) < \Phi(H)$. □

3 Graphs with the maximum number of k-factors

First we introduce some simplifying notation. In the following it is understood that $H = G \cup xy - uv$, for instance, creates the graph H from G by adding the edge xy and removing the edge uv. Also, for conciseness we use the abbreviation $uS = \{uv \mid v \in S\}$.

We also introduce a lemma. The proof of Theorem 3.2 in this section involves an auxiliary bipartite graph, and the following result due to Galvin (appearing as a supplemental exercise in [14]) will be useful.

Lemma 3.1 *Let G be a bipartite graph with partite sets X, Y. Suppose that X has no isolated vertices, and that whenever $xy \in E(G)$ with $x \in X$, $y \in Y$, then $d(x) \geq d(y)$. Then G has a matching that covers X.*

The graph transformation described in the main theorem of this section was independently shown in [5, 13] to not decrease (and typically increase) the number of spanning trees of a graph, and in [12] it was shown that this transformation more generally increases all-terminal network reliability. In the theorem below we show that, parallel to those results, the resulting graph H has at least as many k-factors as G, for any $k \geq 1$.

Theorem 3.2 *Let G be a graph with $x, y \in V(G)$ such that $N_G(y) - x \subset N_G(x) - y$, and let $S \subseteq (N_G(x) - y) - (N_G(y) - x)$. Let $H = G \cup yS - xS$. Then $\Phi_k(G) \leq \Phi_k(H)$ for all $k \geq 1$.*

Proof: Let $k \geq 1$ be fixed. Let F_G denote the set of k-factors of G, and for a set $M \subseteq E(G)$ let $F_G(M)$ denote the set of k-factors F of G for which $M \cap F \neq \emptyset$, i.e. $F_G(M)$ is the set of k-factors that use at least one edge of M. The sets F_G and F_H may be partitioned as $F_G = (F_G - F_G(xS)) \cup F_G(xS)$ and $F_H = (F_H - F_H(yS)) \cup F_H(yS)$, respectively. Since $G - xS = H - yS$, then $F_G - F_G(xS) = F_H - F_H(yS)$, and to show $\Phi_k(G) \leq \Phi_k(H)$ we need only show $|F_G(xS)| \leq |F_H(yS)|$.

To do so we consider the bipartite graph B whose partite sets are $F_G(xS)$ and $F_H(yS)$, with an edge between $F \in F_G(xS)$ and $F' \in F_H(yS)$ if and only if for $T = S \cap N_F(x)$ there exists $U \subseteq N_F(y) - N_F(x)$ such that $F' = F \cup (yT \cup xU) - (xT \cup yU)$. Our goal now is to prove that

(1) For all $F \in F_G(xS)$, we have $d_B(F) \geq 1$.

(2) If $F \in F_G(xS)$ and $F' \in F_H(yS)$ are adjacent in B, then $d_B(F) \geq d_B(F')$.

Once (1) and (2) are shown, Lemma 3.1 guarantees that a matching exists that covers $F_G(xS)$. In particular, $|F_G(xS)| \leq |F_H(yS)|$, completing the proof of the theorem.

Let $F \in F_G(xS)$. By definition of $F_G(xS)$ we have $T = S \cap N_F(x) \neq \emptyset$, and to guarantee $d_B(F) \geq 1$ we only require the existence of a $U \subseteq N_F(y) - N_F(x)$ such that $|U| = |T|$. Since $|N_F(x)| = |N_F(y)| = k$, we have $|N_F(y) - N_F(x)| = |N_F(x) - N_F(y)|$. But $T = S \cap N_F(x) \subseteq N_F(x) - N_F(y)$, and so $|N_F(y) - N_F(x)| \geq |T|$. Thus in fact $d_B(F) = \binom{|N_F(y) - N_F(y)|}{|T|} = \binom{|N_F(x) - N_F(y)|}{|S \cap N_F(x)|}$, and since $S \cap N_F(x) \subseteq N_F(x) - N_F(y)$, in particular $d_B(F) \geq 1$. This proves (1).

Now let $F \in F_G(xS)$ and $F' \in F_H(yS)$ be adjacent in B. By a similar argument as for $d_B(F)$, we see that $d_B(F') = \binom{|(N_{F'}(x) - N_{F'}(y)) \cap N_G(y)|}{|S \cap N_{F'}(y)|}$. Now note that, since $F' = F \cup (yT \cup xU) - (xT \cup yU)$ for some $T = S \cap N_F(xS)$ and $U \subseteq N_F(y) - N_F(x)$, then we have $S \cap N_F(x) = S \cap N_{F'}(y)$ and $N_F(x) \cap$

$N_F(y) = N_{F'}(x) \cap N_{F'}(y)$. Thus

$$\begin{aligned} d_B(F') &= \binom{(N_{F'}(x) - N_{F'}(y)) \cap N_G(y)|}{|S \cap N_{F'}(y)|} \\ &\leq \binom{N_{F'}(x) - N_{F'}(y)|}{|S \cap N_{F'}(y)|} \\ &= \binom{N_F(x) - N_F(y)|}{|S \cap N_F(y)|} \\ &= d_B(F) \end{aligned}$$

This proves (2), and completes the proof of the theorem. □

When $\ell \leq n/2$, we denote the complete graph with a matching of ℓ edges removed by $K_n - \ell K_2$. The fact that, $K_n - \ell K_2$ has at least as many k-factors as any other graph in its class, for any $k \geq 1$, is an immediate consequence of the previous theorem.

Theorem 3.3 *Let* $\ell \leq n/2$, $m \geq \binom{n}{2} - n/2$, *and let* $G \in G_{n,m}$. *Then* $\Phi_k(G) \leq \Phi_k(K_n - \ell K_2)$, *for any* $k \geq 1$.

Proof: In [12], it was shown that $K_n - \ell K_2$ may be obtained from the graph $G \in G_{n,m}$ via a sequence of the graph transformations of Theorem 3.2. The result follows. □

4 Graphs with the minimum number of 1-factors

Graphs with few 1-factors have been studied; for example, for results on graphs with precisely one 1-factor see [6, Ch. 5] and [2, Ch. 2]. Threshold graphs also have been much studied and there are many equivalent ways to define them (see, e.g., [7] or [9, Ch. 5]). For our purposes, it is natural to think of them as split graphs with an additional neighborhood property. A *split graph* is a graph G whose vertex set may be partitioned into two sets $V(G) = A(G) \cup C(G)$, such that $G[A(G)]$ is an independent set and $G[C(G)]$ is a clique. A *threshold graph* is a split graph where the neighborhoods of the vertices of A can be nested with respect to set inclusion. In [12], it was shown that for any n, m, and any graph $H \in G_{n,m}$, there is a threshold graph $G \in G_{n,m}$ such that H can be obtained from G via a sequence of graph transformations from Theorem 3.2. As an immediate consequence of that fact and Theorem 3.2, we have the following.

Theorem 4.1 *Let $k \geq 1$ be fixed. For any $H \in G_{n,m}$, there exists a threshold graph $G \in G_{n,m}$ such that $\Phi_k(G) \leq \Phi_k(H)$.*

Thus when finding a graph with the minimum number of 1-factors, we may restrict our attention to threshold graphs. We mention in passing that in [12] it was also shown that if the graph H is connected, we may take the threshold graph G to be connected. Thus Theorem 4.1 holds with both H and G taken to be connected, as well.

In $G_{n,m}$, of course, finding a threshold graph with the minimum number of 1-factors is trivial for many values of n and m; if $\binom{n}{2} - m \geq n-1$, we may take $G \in G_{n,m}$ to have an isolated vertex and the minimum number of 1-factors G may have is zero. In this section we settle the question for the remaining cases when $m \geq \binom{n}{2} - (n-2)$.

It is easy to determine $\Phi(G)$ exactly for any threshold graph G. In the formula below, $n!!$ is the double factorial, $n!! = n(n-2)(n-4)...$, where we take $(-1)!! = 1$.

Lemma 4.2 *Let n be even, and $G \in G_{n,m}$ be a threshold graph with $A(G) = \{v_1, \ldots, v_j\}$ and $d(v_1) \leq \cdots \leq d(v_j)$. Then*

$$\Phi(G) = (n-2j-1)!! \prod_{i=1}^{j} (d(v_i) - i + 1),$$

if $n \geq 2j$ and $d(v_i) \geq i$ for all $1 \leq i \leq j$; otherwise $\Phi(G) = 0$.

Proof: That we require $n \geq 2j$ and $d(v_i) \geq i$ for all $1 \leq i \leq j$ is obvious. Now consider the ways A may be matched in a perfect matching. Once the vertex v_1 is matched with any of the $d(v_1)$ vertices in its neighborhood, there remain $d(v_2) - 1$ vertices to which x_2 may be matched, which in turn leaves $d(v_3) - 2$ vertices to which x_3 may be matched, and so on. Once the j vertices of A have been matched, there remains a clique of $n - 2j$ vertices of C to be matched, and there are $(n-2j-1)!!$ ways to accomplish this. □

Now as a preliminary step we identify which threshold graphs with fixed values of n, m, and $|A(G)|$ have the fewest 1-factors. We assume from this point forward that $|C(G)|$ is as large as possible, i.e., no vertex in $A(G)$ is adjacent to every vertex of $C(G)$. (If so, then one of those points may be moved to $C(G)$.) Note that as a consequence of this assumption, $d(v) \leq n - |A(G)| - 1$ for all $v \in A(G)$. We also require the following terminology: given two sequences $x = (x_1, \ldots, x_j)$, $y = (y_1, \ldots, y_j)$, we say x *majorizes* y, denoted $x \succ y$, if

$$\sum_{i=1}^{\ell} x_i \geq \sum_{i=1}^{\ell} y_i$$

for all $1 \leq \ell \leq j$, with equality for $\ell = j$. A function $f(x_1, \ldots, x_j)$ is *Schur-convex* if $f(x) \geq f(y)$ whenever x majorizes y.

Lemma 4.3 *Let n be even, $m \geq \binom{n}{2} - (n-2)$, and let $T_j \in G_{n,m}$ be the threshold graph with $A(T_j) = \{v_1, \ldots, v_j\}$ satisfying $d(v_2) = \cdots = d(v_j) = n - j - 1$. Then if $G \in G_{n,m}$ is any other threshold graph with $|A(G)| = j$, we have $\Phi(T_j) \leq \Phi(G)$.*

Proof: Let $A(T_j) = \{v_1, \ldots, v_j\}$ and $A(G) = \{w_1, \ldots, w_j\}$. We may assume that $d(w_1) \leq \cdots \leq d(w_j)$ and $d(w_i) \leq n-j-1$ for all $1 \leq i \leq j$. Since $\sum_{i=1}^{j} d(v_i) = \sum_{i=1}^{j} d(w_i)$, this implies that $d(v_1) < d(w_1)$ and $d(v_i) \geq d(w_i)$ for all $2 \leq i \leq j$. As a consequence,

$$(d(v_2) - 1, \ldots, d(v_j) - j + 1, d(v_1)) \succ (d(w_2) - 1, \ldots, d(w_j) - j + 1, d(w_1)).$$

Call the two sequences above d and d' respectively, so $d \succ d'$. It is well known (see, e.g., [8, Ch. 3.F]) that the function $f(x_1, \ldots, x_j) = -\prod_{i=1}^{j} x_i$ is Schur-convex. Therefore $f(d) \geq f(d')$, or

$$\prod_{i=1}^{j} (d(v_i) - i + 1) \leq \prod_{i=1}^{j} (d(w_i) - i + 1).$$

Since $|A(T_j)| = |A(G)| = j$, multiplying both sides of the inequality by $(n - 2j - 1)!!$ gives $\Phi(T_j) \leq \Phi(G)$, as required. □

As in Lemma 4.3, let $T_j \in G_{n,m}$ denote the threshold graph with $A(G) = \{v_1, \ldots, v_j\}$ and $d(v_2) = \cdots = d(v_j) = n - j - 1$. Then the graphs with the minimum number of 1-factors are the graphs T_1, i.e., the threshold graph with $|A(G)| = 1$ and $|C(G)| = n - 1$.

Theorem 4.4 *Let n be even, $m \geq \binom{n}{2} - (n-2)$. Then $\Phi(T_1) \leq \Phi(G)$ for all $G \in G_{n,m}$.*

Proof: By the previous lemma it suffices to show that $\Phi(T_j) \leq \Phi(T_{j+1})$ for all $j \geq 1$. Let $A(T_j) = \{v_1, \ldots, v_j\}$ and $A(T_{j+1}) = \{w_1, \ldots, w_{j+1}\}$, and note that $d(v_i) = n - j - 1$ for all $2 \leq i \leq j$, and that $d(w_i) = n - j - 2$ for all $2 \leq i \leq j+1$. Since $m(T_j) = m(T_{j+1})$, we have

$$\binom{j}{2} + (n - j - d(v_1)) + j - 1 = \binom{j+1}{2} + (n - (j+1) - d(w_1)) + j,$$

or, after simplification, $d(v_1) + j = d(w_1)$. But it is easy to see that, after canceling like terms,

$$\frac{\Phi(T_j)}{\Phi(T_{j+1})} = \frac{d(v_1)(n-j-2)}{d(w_1)(n-2j-2)} = \frac{d(v_1)(n-j-2)}{(d(v_1)+j)(n-2j-2)}.$$

This fraction is no larger than 1 provided that $d(v_1) \leq n - 2j - 2$. But $d(v_1) + j = d(w_1) \leq d(w_2) = n - j - 2$, completing the proof. $\square$

We conjecture that T_1 has the minimum number of k-factors in its class for all $k \geq 2$ as well.

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