Sufficient Degree Conditions For k-Edge-Connectedness Of A Graph

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Abstract

One of Frank Boesch's best known papers is 'The strongest monotone degree conditions for *n*-connectedness of a graph' [1]. In this paper, we give a simple sufficient degree condition for a graph to be *k*-edge-connected, and also give the strongest monotone condition for a graph to be 2-edge-connected.

KEYWORDS: strongest monotone condition, edge-connectivity, degree sequence, weakly optimal

1 Introduction

Frank Boesch had a long and illustrious career in both electrical engineering and mathematics. However, as those of us who worked with him will tell you, his main love was graph theory. He was especially interested in problems involving the vertex degrees of a graph, and often referred to the well-known algorithm of Havel and Hakimi, which was developed independently by Havel [8] and by Frank's good friend Lou Hakimi [7]. This algorithm allows one to determine if a degree sequence can

^{*}Regretfully, Lou Hakimi died on June 23, 2006. He was a longtime associate editor of *Networks*, and a good friend of Professor Boesch.

be realized as the vertex degrees of a graph, and to effectively construct such a realization when this is the case.

Historically, the degree sequence of a graph has been used to provide sufficient conditions for the graph to have a certain property, e.g. hamiltonian, k-connected, etc. If P denotes a graphical property, we say a graphical degree sequence π is *forcibly* P if every realization of π has property P.

Sufficient conditions for a degree sequence to be forcibly hamiltonian were given by several authors, culminating in the following condition of Chvátal [6] in 1972.

Theorem 1.1. Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If $d_j \leq j < \frac{n}{2}$ implies $d_{n-j} \geq n-j$, then π is forcibly hamiltonian.

Unlike its predecessors, Chvátal's condition has the property that if it does not guarantee that a degree sequence π is forcibly hamiltonian, then π is majorized by a degree sequence π' which has a nonhamiltonian realization. As we will see, this implies that Chvátal's condition is the strongest of an entire class of sufficient conditions for a degree sequence to be forcibly hamiltonian.

Several sufficient conditions for a degree sequence to be forcibly k-connected have also been given. Among the most prominent are the conditions of Chartrand and Harary [4], Chartrand, Kapoor, and Kronk [5], and the following condition of Bondy [2] (though the form in which we give Bondy's condition is due to Boesch [1]).

Theorem 1.2. Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 2$, and let $1 \leq k \leq n-1$. If $d_j \leq j+k-2$ implies $d_{n-k+1} \geq n-j$ for $1 \leq j \leq \lfloor \frac{n-k+1}{2} \rfloor$, then π is forcibly k-connected.

Boesch also showed that if Bondy's condition does not guarantee that π is forcibly *k*-connected, then π is majorized by a sequence π' which has a non-*k*-connected realization. Thus Bondy's condition is a strongest sufficient degree condition for *k*connectedness in precisely the same way that Chvátal's condition is for hamiltonicity.

In the present paper, which is dedicated to Frank Boesch, we develop sufficient conditions for a degree sequence to be forcibly k-edge-connected. Of course, Bondy's condition in Theorem 1.2 provides such a condition. But whereas Theorem 1.2 provides a strongest sufficient condition for k-connectedness, it does not provide a comparably strong condition for k-edge-connectedness. After establishing a general framework in which to identify strongest sufficient degree conditions for a graphical property, we give such a condition for k-edge-connectedness when k = 2. We then conjecture an analogous strongest condition for k = 3, which unfortunately indicates the rapidly increasing complexity of the strongest sufficient condition as k increases. This suggests the desirability of a simple sufficient condition for k-edgeconnectedness which, although not strongest, at least improves Bondy's condition in Theorem 1.2. In the final section, we give such a condition and then compare it to Theorem 1.2. A word about our terminology and notation. In this paper, we consider only undirected graphs without loops or multiple edges, i.e., simple graphs. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms is [9]. A *degree sequence* of a graph on *n* vertices is any sequence $\pi = (d_1, d_2, \ldots, d_n)$ consisting of the vertex degrees of the graph. We will usually assume the degree sequence is in nondecreasing order (in contrast to [9], where degree sequences are usually in nonincreasing order). We will generally use the standard abbreviated notation for degree sequences; e.g., (4, 4, 4, 4, 4, 5, 5) will be denoted 4^55^2 . A sequence of integers $\pi = (d_1, d_2, \ldots, d_n)$ is called *graphical* if there exists a graph *G* having π as one of its vertex degree sequences. In this case we call *G* a *realization* of π . If $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ are two integer sequences, we say that π' majorizes π , denoted $\pi' \geq \pi$, if $d'_j \geq d_j$ for $1 \leq j \leq n$.

2 A Method To Identify Strongest Monotone Degree Conditions

We now describe a framework which unifies and formalizes the techniques first introduced by Chvátal [6], and used by Boesch [1], to establish that Theorems 1.1 and 1.2 are strongest in a certain sense. While this framework can be applied to many graph properties, in the development below we will use k-connectedness as the vehicle property.

Consider a function $f: \{\text{Graphical Degree Sequences}\} \to \{0,1\}$ such that $f(\pi) = 1$ implies π is forcibly k-connected, while $f(\pi) = 0$ implies no conclusion. We call such a function a forcibly k-connected function, since f provides (via $f(\pi) = 1$) a sufficient condition for π to be forcibly k-connected. We call a forcibly k-connected function f monotone increasing if π, π' are graphical, $\pi' \geq \pi$ and $f(\pi) = 1$ implies $f(\pi') = 1$; it is called *optimal* (respectively, *weakly optimal*) if $f(\pi) = 0$ implies π itself has a non-k-connected realization (respectively, there exists $\pi' > \pi$ such that π' has a non-k-connected realization). It is immediate that the forcibly k-connected functions corresponding to the conditions of Chartrand and Harary [4], Chartrand, Kapoor, and Kronk [5], and Bondy [2] are all monotone increasing. Moreover, Boesch [1] proved that the function corresponding to Bondy's condition (Theorem 1.2) is weakly optimal. Indeed, if the condition in Theorem 1.2 fails for some j, $1 \le j \le \lfloor \frac{n-k+1}{2} \rfloor$, then π is majorized by $\pi' = (j+k-2)^j (n-j-1)^{n-j-k+1} (n-1)^{k-j}$, which has a non-k-connected realization $K_{k-1} + (K_j \cup K_{n-j-k+1})$. For brevity in the sequel, we will often abbreviate assertions like 'the forcibly k-connected function corresponding to Theorem 1.2 is weakly optimal' to 'Theorem 1.2 is weakly optimal'.

We now show

Claim 2.1. If f, f_0 are monotone increasing forcibly k-connected functions and f_0 is weakly optimal, then $f_0(\pi) \ge f(\pi)$ for every graphical sequence π .

Proof: Suppose to the contrary that for some graphical sequence π , we have

$$1 = f(\pi) > f_0(\pi) = 0.$$

Since f_0 is weakly optimal, there exists $\pi' \ge \pi$ such that π' has a non-k-connected realization, and thus $f(\pi') = 0$. However $f(\pi) = 1$, $f(\pi') = 0$, and $\pi' \ge \pi$ together imply f is not monotone increasing, a contradiction.

Thus the weakly optimal forcibly k-connected function corresponding to Bondy's condition (Theorem 1.2) is the strongest monotone increasing forcibly k-connected function.

3 Strongest Monotone Degree Condition For 2-Edge-Connectedness

We noted above that Theorem 1.2 provides a sufficient condition for k-edge-connectedness. But unlike the situation for k-connectedness, for which Theorem 1.2 is a strongest monotone condition, Theorem 1.2 is not a strongest monotone condition for k-edge-connectedness, when $k \ge 2$. For instance, let k = 2 and consider $\pi = 2^5$. It is easy to verify that Theorem 1.2 fails to guarantee that π is forcibly 2-edge-connected, but there is no $\pi' \ge \pi$ such that π' has a non-2-edge-connected realization.

We now give the strongest monotone degree condition for 2-edge-connectedness.

Theorem 3.1. Let $\pi = (d_1 \leq \cdots \leq d_n)$ be graphical. If

(1) $d_1 \geq 2$,

(2) $d_{j-1} \leq j-1$ and $d_j \leq j$ implies $d_{n-1} \geq n-j$ or $d_n \geq n-j+1$, for $3 \leq j < \frac{n}{2}$,

(3) n even and $d_{n/2} \leq \frac{n}{2} - 1$ implies $d_{n-2} \geq \frac{n}{2}$ or $d_n \geq \frac{n}{2} + 1$

all hold, then π is forcibly 2-edge-connected.

Proof: It is easy to verify that if π satisfies (1) through (3), then π is forcibly connected by Theorem 1.2. Now suppose π satisfies (1) through (3), but π is not forcibly 2-edge-connected. Then π has a realization G consisting of two connected components H_1 and H_2 of order j and n-j, respectively, joined by a cut-edge (x, y)where the vertices x and y belong to H_1 and H_2 , respectively. We may assume that $3 \leq j \leq \lfloor \frac{n}{2} \rfloor$, since $j \leq 2$ implies $d_1 = 1$, contradicting (1). Each vertex in $H_1 - \{x\}$ has degree at most j - 1, while x has degree at most j. Thus $d_{j-1} \leq j - 1$ and $d_j \leq j$.

If $j < \frac{n}{2}$, each vertex in H_1 has degree at most n - j - 1, each vertex in $H_2 - \{y\}$ has degree at most n - j - 1, and y has degree at most n - j. Thus $d_{n-1} \le n - j - 1$ and $d_n \le n - j$, and so (2) fails for j, a contradiction.

If n is even and $j = \frac{n}{2}$, each vertex in $G - \{x, y\}$ has degree at most $\frac{n}{2} - 1$, while x, y have degree at most $\frac{n}{2}$. Thus $d_{n-2} \leq \frac{n}{2} - 1$ and $d_n \leq \frac{n}{2}$, and the consequent in (3) fails. On the other hand, $n - 1 \geq d_1 \geq 2$ and n even implies $n \geq 4$, and so $\frac{n}{2} \leq n-2$. Thus $d_{n/2} \leq d_{n-2} \leq \frac{n}{2} - 1$, and the antecedent in (3) is satisfied. So

condition (3) fails, a contradiction. This proves Theorem 3.1. \blacksquare

Regarding the weak optimality of Theorem 3.1, we now show that if any of (1), (2), or (3) fail for π , then π is majorized by a degree sequence π' having a realization G' which is not 2-edge-connected. Notationally, let G(n, j) denote two disjoint cliques K_j and K_{n-j} joined by a single cut edge. If (1) fails, we may take $\pi' = 1^1(n-2)^{n-2}(n-1)^1$ and G' = G(n,1). If (2) fails for some $j, 3 \leq j < \frac{n}{2}$, we may take $\pi' = (j-1)^{j-1}j^1(n-j-1)^{n-j-1}(n-j)^1$ and G' = G(n,j). Finally, if (3) fails we may take $\pi' = (\frac{n}{2} - 1)^{n-2}(\frac{n}{2})^2$ and $G' = G(n, \frac{n}{2})$. Thus Theorem 3.1 is weakly optimal.

Ideally, we would like to extend Theorem 3.1 to give a strongest monotone degree condition for a graph to be k-edge-connected for $k \geq 3$. However, this appears to be quite unwieldy. As in Theorem 3.1 for k = 2, we need to find a weakly optimal degree condition that blocks π from having a realization which is a subgraph of a graph consisting of two disjoint cliques K_j and K_{n-j} , $j \leq \lfloor \frac{n}{2} \rfloor$, joined by an edge cut of k - 1 edges. Unfortunately, these edge cuts can range in form from k - 1independent edges to a star, with each different form demanding a different set of blocking conditions. For k = 3, an analysis similar to the one used to develop Theorem 3.1 suggests the following is the strongest monotone sufficient condition.

Conjecture 3.2. Let $\pi = (d_1 \leq \cdots \leq d_n)$ be graphical. If

- (1) $d_1 \geq 3$,
- (2) $d_{j-2} \leq j-1$ and $d_j \leq j$ implies $d_{n-2} \geq n-j$ or $d_n \geq n-j+1$, for $3 \leq j < \frac{n}{2}$,
- (3) $d_{j-1} \leq j-1$ and $d_j \leq j+1$ implies $d_{n-2} \geq n-j$ or $d_n \geq n-j+1$, for $3 \leq j < \frac{n-1}{2}$,
- (4) $d_{j-2} \leq j-1$ and $d_j \leq j$ implies $d_{n-1} \geq n-j$ or $d_n \geq n-j+2$, for $3 \leq j < \frac{n}{2}$,
- (5) *n* even and $d_{n/2} \leq \frac{n}{2} 1$ implies $d_{n-4} \geq \frac{n}{2}$ or $d_n \geq \frac{n}{2} + 1$,
- (6) *n* odd and $d_{(n-3)/2} \leq \frac{n-3}{2}$ implies $d_{n-3} \geq \frac{n+1}{2}$ or $d_n \geq \frac{n+3}{2}$,
- (7) *n* even and $d_{n/2} \leq \frac{n}{2} 1$ implies $d_{n-3} \geq \frac{n}{2}$ or $d_{n-1} \geq \frac{n}{2} + 1$ or $d_n \geq \frac{n}{2} + 2$

all hold, then π is forcibly 3-edge-connected.

It seems likely that the number of individual conditions in the strongest monotone degree condition for a graph to be k-edge-connected (e.g., conjectured to be 7 conditions when k = 3) will increase rapidly with k. This suggests the desirability of finding a simple, though not weakly optimal, sufficient condition for a degree sequence to be forcibly k-edge-connected.

4 A Simple Degree Condition For k-Edge-Connectivity

Of course, Theorem 1.2 provides a simple condition for a degree sequence to be forcibly k-edge-connected. But Theorem 1.2 was designed as a sufficient condition for k-connectedness rather than k-edge-connectedness. We now give another simple condition which, though not weakly optimal, provides a substantial improvement over Theorem 1.2.

Theorem 4.1. Let $\pi = (d_1 \leq \cdots \leq d_n)$ be graphical, and let $k \geq 1$ be an integer. If

(1) $d_1 \ge k$, and

(2) $d_{j-k+1} \leq j-1 \text{ and } d_j \leq j+k-2 \text{ implies } d_n \geq n-j+k-1 \text{ for } k+1 \leq j \leq \lfloor \frac{n}{2} \rfloor$,

then π is forcibly k-edge-connected.

Proof: Suppose π satisfies (1) and (2), but π is not forcibly k-edge-connected. Then π has a realization with a minimum edge-cut of k-1 or fewer edges joining two components H_1 and H_2 , with $|V(H_1)| = j$, $|V(H_2)| = n-j$ for some $j, 1 \le j \le \lfloor \frac{n}{2} \rfloor$. We consider two cases.

Case 1. $1 \le j \le k$. Since $d_1 \ge k$ by (1), we have

$$d_1 + d_2 + \dots + d_j \ge jk.$$

On the other hand, the sum of the degrees in H_1 is at most j(j-1) + k - 1, and so

$$d_1 + d_2 + \dots + d_j \le j(j-1) + k - 1.$$

Combining these inequalities gives $jk \leq j(j-1) + k - 1$, or $(j-1)(k-j) \leq -1$, which contradicts $1 \leq j \leq k$.

Case 2. $k+1 \leq j \leq \lfloor \frac{n}{2} \rfloor$.

At most k-1 vertices in H_1 can have degree larger than j-1, and so $d_{j-(k-1)} \leq j-1$. Also, no vertex in H_1 can have degree larger than (j-1)+(k-1) = j+k-2, and so $d_j \leq j+k-2$. Since $k+1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, condition (2) implies $d_n \geq n-j+k-1$. But no vertex in the realization can have degree larger than (n-j-1)+(k-1) = n-j+k-2, a contradiction.

The following corollary of Theorem 4.1 is well known [3].

Corollary 4.2. Let $\pi = (d_1 \leq \cdots \leq d_n)$ be graphical. If $\delta \geq \lfloor \frac{n}{2} \rfloor$, then π is forcibly δ -edge-connected.

Proof: Set $k = \delta$ in Theorem 4.1. Condition (1) becomes $d_1 \ge k = \delta$, which is true by definition. Moreover, the range in condition (2) becomes $\delta + 1 \le j \le \lfloor \frac{n}{2} \rfloor$. Since

 $\delta \geq \lfloor \frac{n}{2} \rfloor$, this range is empty, and condition (2) vacuously holds.

We now show that Theorem 4.1 is at least as strong as Theorem 1.2 (Bondy's condition), and sometimes significantly better. We first prove the following.

(a) If π satisfies the hypothesis of Theorem 1.2, then π satisfies the hypothesis of Theorem 4.1.

Proof of (a): If π satisfies the hypothesis of Theorem 1.2 for j = 1, we have $d_1 \ge k$, which is (1) in Theorem 4.1.

To show that π satisfies (2) in Theorem 4.1, suppose that $d_{j-k+1} \leq j-1$ and $d_j \leq j+k-2$ for some j, where $k+1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. In particular, $d_{j-k+1} \leq j-1 = (j-k+1)+k-2$. Setting j'=j-k+1, this becomes $d_{j'} \leq j'+k-2$, where $2 \leq j' \leq \lfloor \frac{n-k+1}{2} \rfloor$. So by the hypothesis of Theorem 1.2, we have $d_{n-k+1} \geq n-j' = n-j+k-1$. A fortiori $d_n \geq n-j+k-1$, which is (2) in Theorem 4.1.

Of course, Theorem 4.1 and Theorem 1.2 are equivalent when k = 1. But we have the following.

(b) For every $k \ge 2$, there exists a degree sequence which satisfies the hypothesis of Theorem 4.1, but not of Theorem 1.2.

Proof of (b): Given $k \ge 2$, choose $n \equiv 1 \pmod{4}$ so large that $\frac{n-1}{2} \ge k$, and consider the degree sequence $\pi = \left(\frac{n-1}{2}\right)^n$. Since $\delta = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$, Corollary 4.2 (and of course Theorem 4.1) guarantees that π is forcibly δ -edge-connected, and thus forcibly k-edge-connected since $k \le \frac{n-1}{2} = \delta$.

On the other hand, Theorem 1.2 fails to show π is even forcibly 2-edge-connected, since for $j = \frac{n-1}{2} = \lfloor \frac{n-2+1}{2} \rfloor$, we have $d_{(n-1)/2} \leq \frac{n-1}{2}$ while $d_{n-1} = \frac{n-1}{2} \ngeq n - \left(\frac{n-1}{2}\right)$. Thus π does not satisfy Theorem 1.2 for any $k \geq 2$.

Finally, we observe that Theorem 4.1 is not weakly optimal for k = 2. For example, consider the sequence $\pi = 4^{6}5^{4}$. Theorem 3.1 shows π is forcibly 2-edge-connected. However, Theorem 4.1 fails to show this, since condition (2) fails for j = 5. But if $\pi' \geq \pi$, then π' must be forcibly 2-edge-connected, since the forcibly 2-edge-connected function corresponding to Theorem 3.1 is monotone increasing. Thus Theorem 4.1 is not weakly optimal for k = 2.

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