Best Monotone Degree Bounds for Various Graph Parameters

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Abstract

We identify best monotone degree bounds for the chromatic number and independence number of a graph. These bounds are best in the same sense as Chvátal's hamiltonian degree condition.

1 Terminology and Notation

We consider only undirected graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms is [14]. A *degree sequence* of a graph is any sequence $\pi = (d_1, d_2, \ldots, d_n)$ consisting of the vertex degrees of the graph. We will usually assume the degree sequence is in nondecreasing order (in contrast to [14], where degree sequences are usually in nonincreasing order). We will generally use the standard abbreviated notation for degree sequences, e.g., $(4, 4, 4, 4, 4, 5, 5)$ will be denoted $4^5 5^2$. A sequence of integers $\pi = (d_1, \ldots, d_n)$ is called *graphical* if there exists a graph G having π as

*Regretfully, Lou Hakimi died in June 2006.

one of its vertex degree sequences. In this case we call G a *realization* of π. If $\pi = (d_1, d_2, \ldots, d_n)$ and $\pi' = (d'_1, d'_2, \ldots, d'_n)$ are two integer sequences, we say that π' *majorizes* π, denoted $\pi' \geq \pi$, if $d'_j \geq d_j$ for $1 \leq j \leq n$. Given graphs G and H, we say that G *degree majorizes* H if some degree sequence of G majorizes some degree sequence of H.

Historically, the vertex degrees of a graph have been used to provide sufficient conditions for the graph to have a certain property; e.g., the condition of Chvátal [7] for a graph to be hamiltonian, or the condition of Bondy [4] and Boesch [3] for a graph to be k-connected. Vertex degrees have also been used to provide bounds for certain parameters of the graph; e.g., the upper bound of Welsh and Powell [13] for the chromatic number $\chi(G)$, or the lower bound of Caro [6] and Wei [12] for the independence number $\alpha(G)$. It is this latter use of vertex degrees that will be of interest in this paper.

In the following sections we will identify upper and lower bounds for $\chi(G)$, $\alpha(G)$, and the clique number $\omega(G)$, which are best monotone in the same sense as Chvátal's hamiltonian degree condition. In particular, we will show that the well-known bound of Welsh and Powell is the best monotone upper bound for $\chi(G)$, and that a lesser known bound of Murphy is the best monotone lower bound for $\alpha(G)$. Accordingly, our goal in the remainder of this introduction is to establish a framework, with appropriate terminology, which allows us to identify best upper or lower bounds for a graph parameter in terms of the vertex degrees. In [2], we gave an analogous framework to identify best sufficient degree conditions for a graph to have a certain property; e.g., 2-edge-connected.

Let p denote a graph parameter, e.g., chromatic number, independence number, etc. A function $f : \{\text{Graphical Degree Sequences}\} \rightarrow \mathbb{Z}^+$ such that $p(G) \leq f(\pi)$ (resp., $p(G) \geq f(\pi)$) for every realization G of π is called an *upper* (resp., *lower*) *bound function* for p. An upper or lower bound function f for p is called *optimal* if $f(\pi) = k$ implies π itself has a realization G with $p(G) = k$. A function $f : \{\text{Graphical Degree Sequences}\} \rightarrow \mathbb{Z}^+$ is called *monotone increasing* (resp. *decreasing*) if π, π' graphical and $\pi' \geq \pi$ implies $f(\pi') \geq f(\pi)$ (resp., $f(\pi') \leq f(\pi)$). A monotone increasing upper bound function f, or monotone decreasing lower bound function f, is called *weakly optimal* if $f(\pi) = k$ implies there exists a degree sequence $\pi' \leq \pi$ such that π' has a realization G' with $p(G') = k$. The definition for a monotone increasing lower bound function, or a monotone decreasing upper bound function, to be weakly optimal is exactly the same, except that $\pi' \geq \pi$.

We now show that a monotone upper or lower bound function f for a graph parameter p which is also weakly optimal is the best monotone upper or lower bound function for p. We prove this only for monotone increasing upper bound functions, but the proofs for the remaining cases are completely analogous.

Claim 1.1. *If f_0 (resp., f) is a weakly optimal (resp., an arbitrary) monotone increasing upper bound function for p, then $f_0(\pi) \leq f(\pi)$, for every graphical sequence π.*

Proof: Suppose that for some graphical sequence π that we have $f_0(\pi) > f(\pi)$. Since f_0 is weakly optimal, there exists $\pi' \leq \pi$ such that π' has a realization G' with $p(G') = f_0(\pi)$. Thus $f(\pi') \geq f_0(\pi)$, since f is an upper bound function for p. But $f(\pi') \geq f_0(\pi) > f(\pi)$ and $\pi' \leq \pi$ together imply f is not monotone increasing, a contradiction. ■

2 Best Monotone Upper Bound for $\chi(G)$

We first illustrate how our framework can be used to identify the best monotone upper bound for the chromatic number $\chi(G)$. Two well-known upper bounds for $\chi(G)$ in terms of the vertex degrees are the trivial upper bound $\chi(G) \leq 1+\Delta(G)$, and the following bound of Welsh and Powell [13].

Theorem 2.1. *Let G be a graph with vertex degrees $d_1 \leq \cdots \leq d_n$. Then*

$$\chi(G) \leq 1 + \max_{1 \leq j \leq n} \min\{n-j, d_j\}.$$

It is immediate that the upper bound functions for χ corresponding to the two upper bounds above are each monotone increasing. Moreover, the function corresponding to the upper bound of Welsh and Powell is weakly optimal. Indeed, if the upper bound function h corresponding to Welsh-Powell gives $h(\pi) = k \doteq 1 + \max_{1 \leq j \leq n} \min\{n-j, d_j\}$, then $\pi \geq \pi' = 0^{n-k}(k-1)^k$ and π' is uniquely realizable as $H \doteq (n-k)K_1 \cup K_k$ with $\chi(H) = k$. Thus the weakly optimal upper bound function for χ corresponding to the upper bound of Welsh and Powell (Theorem 2.1) is the best monotone upper bound function for χ, a fact which does not seem to have been noted previously. We conjecture that it is intractable to compute the optimal upper bound function for χ.

3 Best Monotone Lower Bound For $\alpha(G)$

The most prominent lower bound for $\alpha(G)$ in terms of the vertex degrees was given independently by Caro [6] and Wei [12].

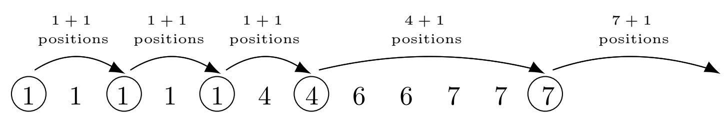
Theorem 3.1. *Let G be a graph with vertex degrees $d_1, d_2, \ldots, d_n$. Then*

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{1+d_i}.$$

An easy induction proof of Theorem 3.1 is outlined in [14, Ex. 3.1.42], and a nice probabilistic proof appears in [1, p. 81].

In [9], Murphy gave an algorithm which yields another lower bound for $\alpha(G)$ in terms of the vertex degrees, and showed this bound was always as good as, and sometimes better than, the bound in Theorem 3.1. As we will see below, Murphy's bound is, in fact, the best monotone lower bound for $\alpha(G)$.

Before formally defining Murphy's bound, we informally illustrate how to obtain the bound for the degree sequence $\pi = 1^5 4^2 6^2 7^3$. Mark the first degree in π (in the diagram below, marked degrees are circled). Thereafter, if a marked degree has value d, move $d+1$ positions to the right in π to reach the next marked degree, and continue until we move beyond the last degree of π.



The resulting number of marked degrees (circles) is Murphy's lower bound for $\alpha(G)$. In the above example, $\alpha(G) \geq 5$ for every realization G of π. By comparison, Theorem 3.1 applied to this degree sequence would guarantee only $\alpha(G) \geq 4$.

We now give a more formal definition of Murphy's bound. Let G be a graph with vertex degrees $d_1 \leq \cdots \leq d_n$. Define a function $d : \mathbb{Z}^+ \to \{d_1, \ldots, d_n, \infty\}$ iteratively as follows: Set $d(1) = d_1$. If $d(j) = d_k$ for some $1 \leq j \leq n$, then

$$\begin{aligned} d(j+1) &= \begin{cases} d_{k+d(j)+1}, & \text{if } k + d(j) + 1 \leq n, \\ \infty, & \text{otherwise} \end{cases} \\ &= \begin{cases} d_{1+\sum_{i=1}^{j}(d(i)+1)}, & \text{if } 1 + \sum_{i=1}^{j}(d(i)+1) \leq n, \\ \infty, & \text{otherwise;} \end{cases} \end{aligned}$$

if $d(j) = \infty$, then $d(j+1) = \infty$.

In the sequence $\pi = 1^5 4^2 6^2 7^3$ above, we have $d(1) = d_1 = 1$, $d(2) = d_3 = 1$, $d(3) = d_5 = 1$, $d(4) = d_7 = 4$, $d(5) = d_{12} = 7$, and $d(6) = d(7) = \cdots = \infty$. Of course, $d(j) \neq \infty$ is the j^{th} marked degree in the informal description above, while $d(j) = \infty$ formally indicates that fewer than j degrees in π are marked before moving past the end of the sequence.

Murphy's lower bound for $\alpha(G)$ is the following.

Theorem 3.2. *Let G be a graph with vertex degrees $d_1 \leq \cdots \leq d_n$. Then*

$$\alpha(G) \geq \max\{j \in \mathbb{Z}^+ \mid d(j) \neq \infty\}.$$

We note that the lower bound in Theorem 3.2 can be arbitrarily larger than the lower bound of Caro and Wei (Theorem 3.1). For instance, consider a graph G with degree sequence $\pi = 1^1 2^2 3^3 \ldots d^d$, where $d \equiv 0 (\mathrm{mod}\ 4)$. We then have the following lower bounds for $\alpha(G)$.

Theorem 3.1:	$\alpha(G) \geq \sum_{j=1}^{d} \left(1 - \frac{1}{j+1}\right) \sim d - \ln d$
Theorem 3.2:	$\alpha(G) \geq d$

Murphy proved Theorem 3.2 by showing inductively that the standard greedy algorithm always produces an independent set of size at least $\max\{j \in \mathbb{Z}^+ \mid d(j) \neq \infty\}$. We now present an alternate proof of Theorem 3.2 that is based on two easy lemmas. The first is a variation of a theorem of Erdös [8]; our proof is modelled on the proof of Theorem 7.8 in [5].

Lemma 3.3. *Let G be a graph with $\alpha(G) \leq r$. Then G degree majorizes a graph H consisting of r disjoint cliques.*

Proof: By induction on r, the result being trivial for $r = 1$.

Choose a vertex x of minimum degree δ in G, and set $G_1 = G - \{x \cup N_G(x)\}$. Since $\alpha(G) \leq r$, we must have $\alpha(G_1) \leq r - 1$. By the induction hypothesis, G_1 degree majorizes a graph H_1 consisting of $r - 1$ disjoint cliques.

Let G_2 denote the complete graph on vertices $\{x \cup N_G(x)\}$. Since each vertex in G_2 has degree δ in $G_1 \cup G_2$, the graph G degree majorizes $G_1 \cup G_2$. A fortiori, G degree majorizes the graph $H \doteq H_1 \cup G_2$, which consists of r disjoint cliques. ■

Lemma 3.4. *If $d(j) \neq \infty$, then $\sum_{i=1}^{j-1}(d(i) + 1) \leq n - 1$.*

Proof: If $d(j) \neq \infty$, then by definition $d(j) = d_{1+\sum_{i=1}^{j-1}(d(i)+1)}$, with $1 + \sum_{i=1}^{j-1}(d(i) + 1) \leq n$, yielding the result. ■

Proof of Theorem 3.2: Let $m = \max\{j \in \mathbb{Z}^+ \mid d(j) \neq \infty\}$, and suppose to the contrary that $\alpha(G) \leq m - 1$, for some realization G of π. By Lemma 3.3, G degree majorizes a graph H consisting of $m - 1$ disjoint cliques.

Suppose $V(H) = X_1 \cup \cdots \cup X_{m-1}$, where each $G[X_i]$ is a clique, and $E(X_i, X_j) = \emptyset$ if $i \neq j$. If we assume $|X_1| \leq \cdots \leq |X_{m-1}|$, the degree sequence π_H of H is

$$\pi_H: \quad \overbrace{|X_1|-1=\cdots=|X_1|-1}^{|X_1|} \leq \overbrace{|X_2|-1=\cdots=|X_2|-1}^{|X_2|} \leq \ldots$$
$$\cdots \leq \overbrace{|X_{m-1}|-1=\cdots=|X_{m-1}|-1}^{|X_{m-1}|}.$$

Immediately below we give the majorizing degree sequence π_G of G, partitioned into $m-1$ 'groups', where the number of degrees in the j^{th} group is $|X_j|$.

$$\pi_G: \quad \underbrace{d_1 \leq \ldots \leq d_{|X_1|}}_{1^{st}\text{ group}} \leq \underbrace{d_{|X_1|+1} \leq \ldots \leq d_{|X_1|+|X_2|}}_{2^{nd}\text{ group}} \leq \cdots$$
$$\cdots \leq \underbrace{d_{|X_1|+\cdots+|X_{m-2}|+1} \leq \ldots \leq d_n}_{(m-1)^{th}\text{ group}}.$$

Since $d(1)=d_1$ majorizes the smallest degree $|X_1|-1$ in π_H, we have $d(1) \geq |X_1|-1$, or $d(1)+1 \geq |X_1|$. Since $d(2)=d_{1+(d(1)+1)}$ and $d(1)+1 \geq |X_1|$, we see that $d(2)$ occurs to the right of the first group in π_G, and thus $d(2) \geq |X_2|-1$, or $d(2)+1 \geq |X_2|$. Continuing in this way, we obtain

$$d(i)+1 \quad \geq \quad |X_i|, \quad \text{for } 1 \leq i \leq m-1. \tag{1}$$

Summing (1) over $1 \leq i \leq m-1$ gives

$$\sum_{i=1}^{m-1}(d(i)+1) \geq \sum_{i=1}^{m-1}|X_i| = V(H) = n.$$

But since $d(m) \neq \infty$, Lemma 3.4 gives $\sum_{i=1}^{m-1}(d(i)+1) \leq n-1$, a contradiction. ■

We now show the lower bound for $\alpha(G)$ in Theorem 3.2 is the best monotone lower bound for $\alpha(G)$. By Claim 1.1, it suffices to show the following.

Theorem 3.5. *The lower bound function for α corresponding to the lower bound in Theorem 3.2 is weakly optimal.*

Proof: Suppose that Theorem 3.2 asserts that $\alpha(G) \geq m \doteq \max\{j \in \mathbb{Z}^+ \mid d(j) \neq \infty\}$. Consider the degree sequence $\pi' = d(1)^{d(1)+1}d(2)^{d(2)+1}\ldots d(m-1)^{d(m-1)+1}\ell^{\ell+1}$, where $\ell \geq 0$ denotes the number of degrees in π to the right of the last marked degree $d(m)$. Note that $\ell \leq d(m)$, since $\ell \geq d(m)+1$ implies that $d(m+1) \neq \infty$, contradicting the definition of

m. It follows immediately that $\pi' \leq \pi$. But π' clearly has a realization $H \doteq K_{d(1)+1} \cup \cdots \cup K_{d(m-1)+1} \cup K_{\ell+1}$ consisting of m disjoint cliques, with $\alpha(H) = m$. Thus the lower bound function for α corresponding to the lower bound in Theorem 3.2 is weakly optimal. ■

We conjecture that it is intractable to compute the optimal lower bound function for α.

4 Best Monotone Lower Bounds For $\omega(G)$ And $\chi(G)$

We now use the best monotone lower bound for $\alpha(G)$ in Theorem 3.2 to obtain best monotone lower bounds for $\omega(G)$ and $\chi(G)$, in terms of the vertex degrees.

Define g : {Graphical Degree Sequences} $\to \mathbb{Z}^+$ by $g(\pi) = f(\overline{\pi})$, where f is the lower bound function for α corresponding to the lower bound for $\alpha(G)$ in Theorem 3.2, and $\overline{\pi} = ((n-1) - d_n \leq \cdots \leq (n-1) - d_1)$ is the complementary degree sequence to $\pi = (d_1 \leq \cdots \leq d_n)$. Then g is a lower bound function for both ω and χ, since $g(\pi) = f(\overline{\pi}) \leq \alpha(\overline{G}) = \omega(G) \leq \chi(G)$. Also, g is monotone increasing, since if $\pi \leq \pi'$, then $\overline{\pi'} \leq \overline{\pi}$ and thus $g(\pi) = f(\overline{\pi}) \leq f(\overline{\pi'}) = g(\pi')$. Thus if we can show that g is a weakly optimal lower bound function for both ω and χ, it will follow that g is the strongest monotone lower bound function for both ω and χ.

Theorem 4.1. *The lower bound function g for ω and χ defined above is weakly optimal.*

Proof: Consider first ω. If $g(\pi) = f(\overline{\pi}) = m$, then by Theorem 3.5 there exists $\overline{\pi'} \leq \overline{\pi}$ such that $\overline{\pi'}$ has a realization $\overline{G'}$ with $\alpha(\overline{G'}) = m$. Since $\overline{\pi'} \leq \overline{\pi}$, we have $\pi \leq \pi'$ and π' has a realization G' with $\omega(G') = \alpha(\overline{G'}) = m$. Thus g is a weakly optimal lower bound function for ω.

Consider next χ. If $g(\pi) = f(\overline{\pi}) = m$, we need to show π is majorized by the degree sequence of a graph with chromatic number m. Consider the degree sequence $\overline{\pi'}$ and realization $\overline{G'}$ in the preceding paragraph. Since $\alpha(\overline{G'}) = m$, Lemma 3.3 implies $\overline{G'}$ degree majorizes a graph $\overline{H}$ consisting of m disjoint cliques. But then H is a complete m-partite graph which degree majorizes G', and of course $\chi(H) = m$. If we let π_H denote the degree sequence of H, then $\pi \leq \pi' \leq \pi_H$. Since π_H has realization H with $\chi(H) = m$, we see that g is a weakly optimal lower bound function for χ as well. ■

We conjecture that it is intractable to compute the optimal lower bound function for either ω or χ.

5 Optimal Upper Bounds For $\alpha(G)$ And $\omega(G)$

In contrast to the conjectured intractability of the optimal bounds mentioned above, it is quite easy to show that computing the optimal upper bound function for either α or ω is tractable. We prove this only for α, since the proof for ω is analogous.

We note first the following variation of a result of Rao [10].

Theorem 5.1. *A degree sequence $\pi = (d_1 \leq \cdots \leq d_n)$ has a realization G with $\alpha(G) \geq k$ if and only if π has a realization in which the vertices with the k smallest degrees form an independent set.*

Using Theorem 5.1, it is easy to determine the largest integer k such that π has a realization G with $\alpha(G) = k$. Iteratively consider the integers $k = 2, 3, 4, \ldots, n$. To decide if π has a realization with k independent vertices, form the graph $H \doteq \overline{K_k} + K_{n-k}$, and let $v_1, \ldots, v_k$ (resp., $v_{k+1}, \ldots, v_n$) denote the vertices of $\overline{K_k}$ (resp., K_{n-k}). Assign d_i to v_i, for $i = 1, 2, \ldots, n$, and determine if H contains a subgraph H' with the assigned degrees. If so, then π has a realization with k independent vertices. Otherwise, by Theorem 5.1, π has no realization with k independent vertices. Tutte [11] has shown the existence of H' is equivalent to the existence of a perfect matching in a graph which can be efficiently constructed from H and $d_1, \ldots, d_n$.

6 Concluding Remarks

We have given best monotone lower bounds for $\alpha(G)$, $\omega(G)$, and $\chi(G)$, and also noted that the upper bound of Welsh and Powell is the best monotone upper bound for $\chi(G)$, in terms of the vertex degrees. We have also conjectured that the optimal lower bound function for α, ω, and χ, as well as the optimal upper bound function for χ, are all intractable. Therefore knowing these best monotone bounds seems particularly important.

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