ON CONSTRUCTING RATIONAL SPANNING TREE EDGE DENSITIES

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Let $\tau(G)$ and $\tau_G(e)$ denote the number of spanning trees of a graph G and the number of spanning trees of G containing edge e of G, respectively. Ferrara, Gould, and Suffel asked if, for every rational $0 < p/q < 1$ there existed a graph G with edge $e \in E(G)$ such that $\tau_G(e)/\tau(G) = p/q$. In this note we provide constructions that show this is indeed the case. Moreover, we show this is true even if we restrict G to claw-free graphs, bipartite graphs, or planar graphs. Let $\text{dep}(G) = \max_{e \in G} \tau_G(e)/\tau(G)$. Ferrara et al. also asked if, for every rational $0 < p/q < 1$ there existed a graph G with $\text{dep}(G) = p/q$. For the claw-free construction, we are also able to answer this question in the affirmative.

1. Introduction

The number of spanning trees of a graph G, denoted $\tau(G)$, has been a well-studied graph parameter since at least 1889, when Cayley first determined that $\tau(K_r) = r^{r-2}$ [3]. Since Cayley's result, formulas for $t(G)$ have been determined for a wide array of graphs [cites]. Often called the *complexity of* G, $\tau(G)$ also plays an important role in a number of graph theory applications, with network reliability and physics (see for instance [5, 6, 16, 18, 21]) perhaps the two most prominent. For a graph G and edge $e \in E(G)$, let $\tau_G(e)$ denote the number of spanning trees of G which contain e. The ratio $d_G(e) = \tau_G(e)/\tau(G)$, the proportion of spanning trees of G which contain e, has also appeared in a number of applications, beginning at least as far back as Kirchhoff [9] where it represents the effective resistance between two adjacent vertices in the graph of an electrical circuit. In theoretical chemistry, $d_G(uv)$ for an edge $uv \in E(G)$ is the *resistance distance* $\Omega(u, v)$, an intrinsic graph metric [10] which has recently developed its own substantial literature (see for instance [2, 4, 13, 20, 22]). The ratio $\tau_G(e)/\tau(G)$ can be interpreted as the probability that a randomly chosen spanning tree contains the edge e, and in this probabilistic context it has been investigated in the context of random walks on graphs [7] and uniform spanning trees [15]. The papers mentioned here are just the 'tip of the iceberg'; the interested reader may consult the references in the papers to learn more.

In [8], Ferrara, Suffel and Gould raised a new and natural question about $d_G(e)$. They called $d_G(e)$ the *spanning tree edge density of* e, a term which for conformance we will use also, and looked at problems involving the *spanning tree edge dependence of* G, or $\text{dep}(G) = \max_{e \in G} d_G(e)$. In network reliability terms, an edge e with $d_G(e) = \text{dep}(G)$ is an edge whose removal would damage the network the most. Let $p/q < 1$ be a positive rational number. If there exists a graph G with edge $e \in E(G)$ such that $d_G(e) = p/q$, we say that the spanning tree edge density p/q is *constructible*. Similarly, if there exists a graph G with $\text{dep}(G) = p/q$, we say that the spanning tree edge dependence is constructible. In [8] the following realizability questions were raised.

Question. Which rational spanning tree edge densities are constructible? More specifically, which spanning tree edge dependencies are constructible?

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In this note we provide graph families that prove that all rational densities $0 < p/q < 1$, i.e., all possible rational densities, are constructible. Moreover, we are able to show that all such densities are constructible even when restricted to classes of graphs such as claw-free graphs, bipartite graphs, or planar graphs. For the claw-free construction, we are also able to answer the second question in the affirmative, showing that all spanning tree edge dependencies are constructible.

In the rest of the paper we will almost invariably shorten the terms 'spanning tree edge density' and 'spanning tree edge dependence' to 'density' and 'dependence', respectively. We also mention the following terminology, which will be useful. A spanning tree is a spanning forest with one component; a closely related graph structure, the *spanning thicket* [1] (sometimes called a *spanning bitree* [15]) is a spanning forest with exactly two components. If vertices u, v lie in different components of a spanning thicket, we say that the thicket *separates* u *and* v, or that u and v are *separated* by the thicket. We denote the number of spanning thickets of a graph G by $b(G)$, and the number of spanning thickets of G that separate vertices u and v by $b_G(u, v)$. A good reference for any terms not defined in the body of the paper is [17].

2. Necklace Graphs

Informally, our first construction is obtained by replacing the edges of a cycle with graphs, resulting in a sequence of graphs "glued" together in a circle. We will also require that, after the replacements, the edges of the original cycle have been replaced by edges of the replacement graphs; these edges will play a key role in the construction of the required densities. Formally, let $\{G_i\}_{i=1}^n$ be a sequence of graphs, each with a distinguished edge $e_i = u_i v_i$. The *necklace graph* $N(G_1(e_1), \ldots, G_n(e_n))$ is obtained by identifying each vertex v_i with each u_{i+1}, where the indices are taken mod n. When there is no confusion we may omit the distinguished edges and write $N(G_1, \ldots, G_n)$. For reference, we will refer to the individual G_i as the *constituent graphs* of the necklace graph, the distinguished (cycle) edges $e_1, \ldots, e_n$ will be the *key edges* and the vertices u_i, v_i for which $e_i = u_i v_i$ will be the *key vertices* of the necklace graph.

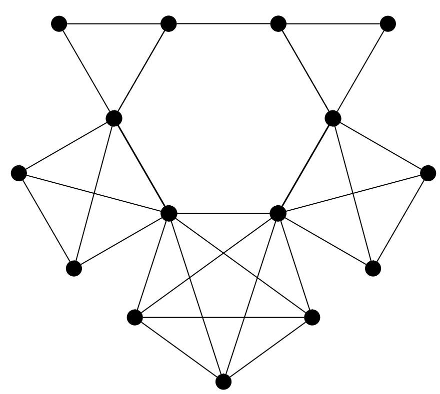

Figure 1. The necklace graph $G = N(K_2, K_3, K_4, K_5, K_4, K_3)$.

Given a necklace graph G, we can calculate the quantities $\tau(G)$ and $\tau_G(xy)$ for an arbitrary edge $xy \in E(G)$ in terms of the spanning trees and thickets of the constituent graphs.

Theorem 2.1. *Let $G = N(G_1, \ldots, G_n)$ be a necklace graph. Then*

$$\tau(G) = \prod_{i=1}^{n} \tau(G_i) \sum_{i=1}^{n} d_{G_i}(u_i v_i)$$

and, for any $xy \in E(G_k)$, $1 \leq k \leq n$,

$$\tau_G(xy) = \prod_{i=1}^{n} \tau(G_i) \left(\frac{b_{G_k}(xy; u_k, v_k)}{\tau(G_k)} + d_{G_k}(xy) \sum_{i \neq k} d_{G_i}(u_i v_i) \right)$$

where $b_{G_k}(xy; u_k, v_k)$ *denotes the number of spanning thickets of* G_k *separating* u_k *and* v_k *that contain the edge* xy.

Proof. Any spanning tree of G is formed by selecting a spanning thicket of a particular G_j, $1 \leq j \leq n$, that separates u_j and v_j, and taking spanning trees of the remaining G_i, $i \neq j$. Summing over all j we obtain

$$\begin{aligned} \tau(G) &= \sum_{j=1}^{n} \left(b_{G_j}(u_j, v_j) \prod_{i \neq j} \tau(G_i) \right) \\ &= \prod_{i=1}^{n} \tau(G_i) \sum_{j=1}^{n} \frac{b_{G_j}(u_j, v_j)}{\tau(G_j)}. \end{aligned}$$

It is easy to see that adding or removing the edge $u_j v_j$ produces a bijection between the spanning thickets of G_j that separate u_j and v_j, and the spanning trees of G_j that contain the edge $u_j v_j$. Thus $b_{G_j}(u_j, v_j) = \tau_{G_j}(u_j v_j)$, and the above becomes

$$\begin{aligned} \tau(G) &= \prod_{i=1}^{n} \tau(G_i) \sum_{j=1}^{n} \frac{\tau_{G_j}(u_j v_j)}{\tau(G_j)} \\ &= \prod_{i=1}^{n} \tau(G_i) \sum_{i=1}^{n} d_{G_i}(u_i v_i). \end{aligned}$$

proving the first part of the theorem.

If $xy \in E(G_k)$ is in a spanning tree of necklace graph G, then either xy is part of a spanning thicket of G_k or xy is part of a spanning tree of G_k. Thus, by a similar analysis as in the calculation of $\tau(G)$, we have

$$\begin{aligned} \tau_G(xy) &= b_{G_k}(xy; u_k, v_k) \prod_{i \neq k} \tau(G_i) + \tau_{G_k}(xy) \sum_{i \neq k} \left(b_{G_i}(u_i, v_i) \prod_{j \neq i,k} \tau(G_j) \right) \\ &= \prod_{i=1}^{n} \tau(G_i) \left(\frac{b_{G_k}(xy; u_i, v_i)}{\tau(G_k)} + \frac{\tau_{G_k}(xy)}{\tau(G_k)} \sum_{i \neq k} \frac{b_{G_i}(u_i, v_i)}{\tau(G_i)} \right) \\ &= \prod_{i=1}^{n} \tau(G_i) \left(\frac{b_{G_k}(xy; u_i, v_i)}{\tau(G_k)} + d_{G_k}(xy) \sum_{i \neq k} \frac{t_{G_i}(u_i v_i)}{\tau(G_i)} \right) \\ &= \prod_{i=1}^{n} \tau(G_i) \left(\frac{b_{G_k}(xy; u_i, v_i)}{\tau(G_k)} + d_{G_k}(xy) \sum_{i \neq k} d_{G_i}(u_i v_i) \right) \end{aligned}$$

□

As a consequence of Theorem 2.1 we have the following result on key edges.

Theorem 2.2. *Let $G = N(G_1, \ldots, G_n)$ be a necklace graph. Then for any key edge $e_k = u_k v_k$, $1 \le k \le n$,*

$$d_G(u_k v_k) = d_{G_k}(u_k v_k) \left(\frac{\sum_{i \ne k} d_{G_i}(u_i v_i)}{\sum_{i=1}^{n} d_{G_i}(u_i v_i)} \right) = d_{G_k}(u_k v_k) \left(1 - \frac{d_{G_k}(u_k v_k)}{\sum_{i=1}^{n} d_{G_i}(u_i v_i)} \right).$$

Proof. If an edge $e_k = u_k v_k$ appears in a spanning thicket then that thicket cannot separate u_k and v_k. In particular, $b_{G_k}(u_k v_k; u_k, v_k) = 0$. Applying this fact in the previous theorem, we obtain

$$\tau_G(u_k v_k) = \left(\prod_{i=1}^{n} \tau(G_i) \right) d_{G_k}(u_k v_k) \sum_{i \ne k} d_{G_i}(u_i v_i)$$

for any key edge $u_k v_k \in E(G_k)$. The formula for $d_G(u_k v_k)$ now follows easily. □

Note that the density of any key edge in the necklace graph depends only upon the densities of the key edges within the constituent graphs $G_1, \ldots, G_n$.

3. Constructing Densities

In this section we show that any rational density $0 < p/q < 1$ is constructible by a claw-free graph G, bipartite graph G, or planar graph G. The constructions all take advantage of properties of unit fractions (sometimes called Egyptian fractions), rational numbers of the form $1/t$ for a positive integer t. We gather the needed properties together in the following simple lemma, a proof of which we include for completeness.

Lemma 3.1. *Any rational number may be written as the sum of unit fractions. More specifically, any rational number may be written as (a) the sum of an even number of unit fractions, and (b) the sum of an arbitrarily large number of unit fractions.*

Proof. A rational number p/q can always be written as the sum of p terms, each of which is $1/q$. And if we take any sum of unit fractions then replacing any specific term in the sum, say $1/t$, with the two unit fractions $1/2t$ and $1/2t$ does not change the sum but increases the number of unit fractions in the sum by one. This fact proves both (a) and (b). □

In the constructions of this section we will use edge-transitive graphs as the constituent graphs of our necklace graphs. For these graphs calculating edge densities is straightforward.

Theorem 3.2 ([8, 10])**.** *Let G be an edge-transitive graph. Then for any edge $e \in E(G)$,*

$$d_G(e) = \frac{|V(G)| - 1}{|E(G)|}.$$

(In a coincidence, the quantity $|E(G)|/(|V(G) - 1|)$ has been called the "density" of a graph in the context of disjoint spanning trees, see for instance [11, 12].)

We can now construct any rational edge density. By Lemma 3.1, condition (1) in the following theorem may always be satisfied.

Theorem 3.3. *Let p, q be positive integers, $p < q$, and let $G = N(K_{r_1}, \ldots, K_{r_n})$ with $r_1 = 2$ and $r_i > 1$, $2 \le i \le n$ such that*

$$\sum_{i=2}^{n} \frac{1}{r_i} = \frac{p}{2(q-p)}. \tag{1}$$

Then $d_G(e_1) = p/q$.

Proof. Since a complete graph K_r is edge-transitive then by Theorem 3.2, the density of any edge is

$$d_{K_r}(e) = \frac{r-1}{r(r-1)/2} = \frac{2}{r}.$$

In particular, since $G_1 = K_2$ we have $d_{G_1}(e_1) = 1$ and $d_{G_i}(u_i v_i) = 2/r_i$ for $2 \leq i \leq n$. Thus, by Theorem 2.2, the density of e_1 in G is

$$d_G(e_1) = d_{G_1}(e_1)\left(\frac{\sum_{i=2}^n d_{G_i}(u_i v_i)}{\sum_{i=1}^n d_{G_i}(u_i v_i)}\right) = \frac{\sum_{i=2}^n \frac{2}{r_i}}{1+\sum_{i=2}^n \frac{2}{r_i}} = \frac{p/(q-p)}{1+p/(q-p)} = \frac{p}{q}$$

as desired. □

In the next section we will show that, if the r_i's for $2 \leq i \leq n$ in condition (1) are chosen to be large enough, the density $d_G(e_1)$ will in fact be the maximum density of the given necklace graph. Hence the construction of Theorem 3.3 will also show that every rational spanning tree edge dependence is constructible.

It is easy to see that the necklace graph of Theorem 3.3 is claw-free, and so we note the following.

Corollary 3.4. *Let p, q be positive integers, $p < q$. Then there exists a claw-free graph G with edge e such that $d_G(e) = p/q$.*

A similar construction, which uses complete bipartite graphs in place of complete graphs, demonstrates that densities are also constructible by bipartite graphs.

Theorem 3.5. *Let p, q be positive integers, $p < q$, let t_i, $2 \leq i \leq n$, be positive integers such that*

$$\sum_{i=2}^n \frac{1}{t_i} = \frac{p}{q-p}.$$

Let $r_1 = s_1 = 1$ and, for all $2 \leq i \leq n$, let $r_i = 2t_i$ and $s_i = 2t_i - 1$. Then in $G = N(K_{r_1,s_1}, K_{r_2,s_2}, \ldots, K_{r_n,s_n})$ we have $d_G(e_1) = p/q$.

Proof. Since a complete bipartite graph $K_{r,s}$ is edge-transitive then the density of any edge is

$$d_{K_{r,s}}(e) = \frac{r+s-1}{rs}.$$

In particular, since $G_1 = K_{1,1}$ then $d_{G_1}(u_1 v_1) = 1$, and for $2 \leq i \leq n$,

$$d_{G_i}(u_i v_i) = \frac{r_i + s_i - 1}{r_i s_i} = \frac{2t_i + (2t_i - 1) - 1}{2t_i(2t_i - 1)} = \frac{4t_i - 2}{2t_i(2t_i-1)} = \frac{1}{t_i}.$$

Thus

$$d_G(e_1) = d_{G_1}(u_1 v_1)\left(\frac{\sum_{i=2}^n d_{G_i}(u_i v_i)}{\sum_{i=1}^n d_{G_i}(u_i v_i)}\right) = \frac{\sum_{i=2}^n \frac{r_i+s_i-1}{r_i s_i}}{1+\sum_{i=2}^n \frac{r_i+s_i-1}{r_i s_i}} = \frac{\sum_{i=2}^n \frac{1}{t_i}}{1+\sum_{i=2}^n \frac{1}{t_i}} = \frac{p/(q-p)}{1+p/(q-p)} = \frac{p}{q}$$

as desired. □

Since the constituent graphs of the necklace graph of Theorem 3.5 have no odd cycles, then in order to insure no odd cycles exist in the necklace graph we need only insure that the conditions of the theorem can be satisfied with n even. By Lemma 3.1(a), this is always possible, so we have the following.

Corollary 3.6. *Let p, q be positive integers, $p < q$. Then there exists a bipartite graph G with edge e such that $d_G(e) = p/q$.*

A different construction can produce rational densities via a planar graph, again by utilizing sums of unit fractions. A *generalized theta graph* $\Theta(r_1, \ldots, r_n)$ is a graph consisting of two distinguished vertices u, v with n disjoint paths between them, of lengths (in edges) of $r_1, \ldots, r_n$. (When $n = 2$ we obtain a cycle with $r_1 + r_2$ edges. A generalized theta graph with $n = 3$ is typically called simply a theta graph.) Note that u and v are adjacent in $\Theta(r_1, \ldots, r_n)$ if and only if $r_k = 1$ for some $1 \leq k \leq n$. We now have the following.

Theorem 3.7. *Let $G = \Theta(r_1, \ldots, r_n)$ be a generalized theta graph. Then*

$$\tau(G) = \sum_{j=1}^{n} \prod_{i \neq j} r_i = \prod_{i=1}^{n} r_i \left(\sum_{i=1}^{n} \frac{1}{r_i} \right)$$

and, if e_k is an edge of the k^{th} path of G, then

$$\tau_G(e_k) = \prod_{i=1}^{n} r_i \left(\sum_{i=1}^{n} \frac{1}{r_i} - \frac{1}{r_k} \sum_{i \neq k} \frac{1}{r_i} \right).$$

Proof. A spanning tree is obtained from G by removing an edge from all but one of the n paths joining u and v. Hence

$$\tau(G) = \sum_{j=1}^{n} \prod_{i \neq j} r_i = \prod_{i=1}^{n} r_i \left(\sum_{i=1}^{n} \frac{1}{r_i} \right).$$

To determine $\tau_G(e_k)$ where e_k is an edge of the k^{th} path, we note that the number of spanning trees that do not contain e_k is $\tau(G - e_k)$, and this is equal to the number of spanning trees of the multipath graph G with the k^{th} path removed. Hence

$$\begin{aligned} \tau_G(e_k) &= \tau(G) - \tau(G - e_k) \\ &= \prod_{i=1}^{n} r_i \left(\sum_{i=1}^{n} \frac{1}{r_i} \right) - \prod_{i \neq k} r_i \left(\sum_{i \neq k} \frac{1}{r_i} \right) \\ &= \prod_{i=1}^{n} r_i \left(\sum_{i=1}^{n} \frac{1}{r_i} - \frac{1}{r_k} \sum_{i \neq k} \frac{1}{r_i} \right). \end{aligned}$$

□

(We note in passing that when $n = t$ and $r_1 = \cdots = r_t = m - 1$, the generalized theta graph obtained is the graph $F(t)$ from [19], in which one result is a formula for $\tau(F(t))$ obtained from a complicated recursion. Theorem 3.7 generalizes that result, and in a much simpler fashion.)

Theorem 3.8. *Let $G = \Theta(1, r_2, \ldots, r_n)$, with*

$$\sum_{i=2}^{n} \frac{1}{r_i} = \frac{q - p}{p}.$$

Then $d_G(uv) = p/q$.

Proof. Since $r_1 = 1$, then by the previous theorem,

$$\tau(G) = \prod_{i=2}^{n} r_i \left(1 + \sum_{i=2}^{n} \frac{1}{r_i} \right) \quad \text{and} \quad \tau_G(uv) = \prod_{i=2}^{n} r_i.$$

Hence

$$d_G(uv) = \frac{\tau_G(uv)}{\tau(G)} = \frac{1}{1+\sum_{i=2}^{n}\frac{1}{r_i}} = \frac{1}{1+(q-p)/p} = \frac{p}{q}$$

□

Since generalized theta graphs are clearly planar, we have the following.

Corollary 3.9. *Let p, q be positive integers, $p < q$. Then there exists a planar graph G with edge e such that $d_G(e) = p/q$.*

4. Constructing Dependencies

In this section we show that the construction of Theorem 3.3, the necklace graph $N(K_2, K_{r_2}, \dots, K_{r_n})$ for specifically chosen $r_2, \dots, r_n$, can produce any desired dependence if the r_i, $2 \le i \le n$, are chosen to be large enough. As before, the edge e_1 (the only edge of the first constituent graph K_2) will carry the required dependence. We proceed by first establishing two lemmas, which consider key edge densities and non-key edge densities of the graph respectively.

Lemma 4.1. *Let $G = N(G_1, \dots, G_n)$ be a necklace graph with key edges $e_i \in E(G_i)$, $1 \le i \le n$. Then $d_G(e_k) \le d_G(e_\ell)$ if and only if $d_{G_k}(e_k) \le d_{G_\ell}(e_\ell)$.*

Proof. Note that

$$(d_{G_k}(e_k) - d_{G_\ell}(e_\ell)) \sum_{i=1}^{n} d_{G_i}(e_i) \le (d_{G_k}(e_k) - d_{G_\ell}(e_\ell))\,(d_{G_k}(e_k) + d_{G_\ell}(e_\ell))$$

if and only if $d_{G_k}(e_k) \le d_{G_\ell}(e_\ell)$. The above is algebraically equivalent to

$$d_{G_k}(e_k)\left(1 - \frac{d_{G_k}(e_k)}{\sum_{i=1}^{n} d_{G_i}(e_i)}\right) \le d_{G_\ell}(e_\ell)\left(1 - \frac{d_{G_\ell}(e_\ell)}{\sum_{i=1}^{n} d_{G_i}(e_i)}\right)$$

which, by Theorem 2.2, says $d_G(e_k) \le d_G(e_\ell)$. □

We divide non-key edges into two classes: those incident to the key vertices of their constituent graphs, and those not incident to the key vertices of their constituent graphs. For brevity, we will refer to these edges as *type 1* and *type 2 edges*, respectively.

We will need the following result, due to Moon [14].

Theorem 4.2 ((Moon, [14])). *Let F be a forest of K_r. Then if $\ell(F)$ denotes the number of components of F, and if $p(F_n)$ denotes the product of the number of vertices in the $\ell(F)$ components of F, we have*

$$\tau_{K_r}(F) = p(F)n^{\ell(F)-2}$$

where $\tau_{K_r}(F)$ denotes the number of trees of K_r that contain all of the edges of F.

In particular, note that if $F = P_3 \cup (n-3)K_1$, then $\tau_{K_r}(F) = 3r^{r-4}$, and if $F = 2K_2 \cup (n-4)K_1$, then $\tau_{K_r}(F) = 4r^{r-4}$.

Lemma 4.3. *Let $G = N(K_2, K_{r_2}, \dots, K_{r_n})$. If $xy \in E(K_{r_k})$ is a type 1 edge of G, then*

$$\tau_G(xy) = \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{4}{r_k}\sum_{i=1}^{n}\frac{1}{r_i} - \frac{1}{r_k^2}\right).$$

If $xy \in E(K_{r_k})$ is a type 2 edge of G, then

$$\tau_G(xy) = \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{4}{r_k}\sum_{i=1}^{n}\frac{1}{r_i}\right).$$

Consequently, type 1 edges of the necklace graph $N(K_2, K_{r_2}, \dots, K_{r_k})$ never carry the spanning tree edge dependence of G.

Proof. Let $xy \in E(K_{r_k})$ be a non-key edge of G, with $u_kv_k \in E(K_{r_k})$ the corresponding key edge. It is immediate that the same bijection as in Theorem 2.1 (adding/removing edge u_kv_k) is a bijection between the spanning thickets of K_{r_k} that separate vertices u_k, v_k and also contain edge x_ky_k, and the spanning trees of K_{r_k} which contain edges u_kv_k and x_ky_k. Hence $b_{G_k}(x_ky_k; u_k, v_k) = \tau_{K_{r_k}}(u_kv_k, x_ky_k)$, and by Theorem 2.1,

$$\tau_G(x_ky_k) = \prod_{i=1}^{n}\tau(K_{r_i})\left(\frac{\tau_{K_{r_k}}(x_ky_k, u_kv_k)}{\tau(K_{r_k})} + d_{K_{r_k}}(x_ky_k)\sum_{i\neq k} d_{K_{r_i}}(u_iv_i)\right).$$

Now consider a type 1 edge of G, i.e., an edge $u_kx_k \in E(K_{r_k})$, with u_k a key vertex of K_{r_k} and $x_k \neq v_k$. By Moon's Theorem 4.2, we have $\tau_{K_{r_k}}(u_kv_k, u_kx_k) = 3r_k^{r_k-4}$. By Cayley's formula, $\tau(K_{r_i}) = r_i^{r_i-2}$ and, as in Theorem 3.3, $d_{K_{r_i}}(u_iv_i) = 2/r_i$. We obtain

$$\begin{aligned}
\tau_G(u_kx_k) &= \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{3r_k^{r_k-4}}{r_k^{r_k-2}} + \frac{2}{r_k}\sum_{i\neq k}\frac{2}{r_i}\right)\\
&= \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{3}{r_k^2} + \frac{4}{r_k}\left(\sum_{i=1}^{n}\frac{1}{r_i} - \frac{1}{r_k}\right)\right)\\
&= \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{4}{r_k}\sum_{i=1}^{n}\frac{1}{r_i} - \frac{1}{r_k^2}\right)
\end{aligned}$$

Now consider a type 2 edge of G, i.e., an edge $x_ky_k \in E(K_{r_k})$, with x_k, y_k both distinct from key vertices u_k, v_k. Now by Moon's Theorem 4.2, we have $\tau_{K_{r_k}}(u_kv_k, x_ky_k) = 4r_k^{r_k-4}$, and we obtain

$$\begin{aligned}
\tau_G(x_ky_k) &= \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{4r_k^{r_k-4}}{r_k^{r_k-2}} + \frac{2}{r_k}\sum_{i\neq k}\frac{2}{r_i}\right)\\
&= \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{4}{r_k^2} + \frac{4}{r_k}\left(\sum_{i=1}^{n}\frac{1}{r_i} - \frac{1}{r_k}\right)\right)\\
&= \prod_{i=1}^{n} r_i^{r_i-2}\left(\frac{4}{r_k}\sum_{i=1}^{n}\frac{1}{r_i}\right)
\end{aligned}$$

In particular, for any $2 \le k \le n$, and for x_k, y_k distinct from key vertices u_k, v_k,

$$\tau_G(x_ky_k) - \tau_G(u_kx_k) = \frac{1}{r_k^2}\left(\prod_{i=1}^{n} r_i^{r_i-2}\right) > 0$$

and so the type 1 edges u_kx_k are contained in strictly fewer spanning trees than the type 2 edges x_ky_k. ∎

We are now ready to construct any rational edge dependence. By Lemma 3.1(b), condition (2) in the following theorem may always be satisfied.

Theorem 4.4. *Let p, q be positive integers, $p < q$, and let $G = N(K_2, K_{r_2}, \ldots, K_{r_n})$ such that*

$$\sum_{i=2}^{n} \frac{1}{r_i} = \frac{p}{2(q-p)} \quad \text{and} \quad r_i \geq \frac{2(2q-p)}{p} \tag{2}$$

for all $2 \leq i \leq n$. Then dep$(G) = d_G(e_1) = p/q$.

Proof. It is easy to see that $p < q$ and condition (2) above insure that $r_i \geq 2 > 1$ for all $2 \leq i \leq n$. Thus by Theorem 3.3 we have $d_G(e_1) = p/q$. We now show that $\max_{e \in E(G)} d_G(e) = d_G(e_1)$. By Lemma 4.1 and the construction of G, we know that $d_G(e_1) \geq d_G(e_k)$ for any key edge e_k, $2 \leq k \leq n$. And by Lemma 4.3 we know no type 1 edge carries the dependence. So we need only show that e_1 is in more spanning trees of G than any type 2 edge of G. But for any type 2 edge $x_k y_k \in E(K_{r_k})$, $2 \leq k \leq n$, we have, by Theorem 2.2 and Lemma 4.3,

$$\begin{aligned}
\tau_G(e_1) - \tau_G(x_k y_k) &= \prod_{i=1}^{n} r_i^{r_i-2} \left(\sum_{i=2}^{n} \frac{2}{r_i} \right) - \prod_{i=1}^{n} r_i^{r_i-2} \left(\frac{4}{r_k} \sum_{i=1}^{n} \frac{1}{r_i} \right) \\
&= \prod_{i=1}^{n} r_i^{r_i-2} \left[2 \sum_{i=2}^{n} \frac{1}{r_i} - \frac{4}{r_k} \left(1 + \sum_{i=2}^{n} \frac{1}{r_i} \right) \right] \\
&= \left(\prod_{i=1}^{n} r_i^{r_i-2} \right) \left[2 \left(\frac{p}{2(q-p)} \right) - \frac{4}{r_k} \left(1 + \frac{p}{2(q-p)} \right) \right] \\
&\geq \left(\prod_{i=1}^{n} r_i^{r_i-2} \right) \left[\frac{p}{q-p} - \frac{4}{2(2q-p)/p} \left(1 + \frac{p}{2(q-p)} \right) \right] \\
&= 0.
\end{aligned}$$

So for any type 2 edge $x_k y_k \in E(K_{r_k})$, $2 \leq k \leq n$, we see that $\tau_G(e_1) \geq \tau_G(x_k y_k)$, implying $d_G(e_1) \geq d_G(x_k y_k)$ for any type 2 edge, which completes the proof. □

5. Conclusion and Open Questions

We conclude with two conjectures and two questions which stem from the bipartite and planar constructions of Theorems 3.5 and 3.8. First we conjecture that the bipartite construction of Theorem 3.5 can be made to carry the correct spanning tree edge dependence, similar to the way the complete graph construction of Theorem 3.3 was in section 4.

Conjecture 1. Let p, q be positive integers, $p < q$. There exists some function $f(p, q)$ such that, if G is the bipartite construction of Theorem 3.5, then $t_i \geq f(p, q)$ for all $2 \leq i \leq n$ implies that dep$(G) = d_G(e_1)$.

While it is possible that Conjecture 2 could be proved with less, obviously a bipartite version of Moon's theorem on spanning trees in complete graphs (Theorem 4.2) would be useful. To our knowledge, no such result is known. Hence a more general question motivated by Theorem 3.5 is the following.

Question 1. Is there a complete bipartite version of Theorem 4.2?

In contrast to the bipartite case, it is not difficult to see that in the generalized theta graph of Theorem 3.8 the edge $e = uv$ in fact carries the minimum, not maximum, density in G. At this time, we do not know if it is possible to produce arbitrary spanning tree edge dependencies via planar graphs, although we conjecture it can be done.

Conjecture 2. Let p, q be positive integers, $p < q$. There exists a planar graph G such that $\mathrm{dep}(G) = p/q$.

The above conjecture raises a somewhat more general question, however. As mentioned in the Introduction the resistance distance $\Omega(u, v)$ is equal to $d_G(uv)$ when u, v are adjacent in G. However, $\Omega(u, v)$ is in fact a distance metric between any two vertices in a graph [10], not just adjacent vertices. It is well-known that if u, v non-adjacent then

$$\Omega(u, v) = \frac{\tau_{G+uv}(uv)}{\tau(G)}.$$

In analogy with spanning tree edge dependence, then, we define $\mathrm{rd}(G) = \max_{u,v\in V(G)} \Omega(u, v)$.

Let p/q be any positive rational number. (When vertices are not adjacent, the resistance distance between those vertices can be greater than 1.) It is possible to construct a graph with $\mathrm{rd}(G) = p/q$; indeed a particular necklace graph "with the clasp removed" is such a construction. Specifically, let $G = N(K_2, K_{r_2}, \ldots, K_{r_p})$, with $r_2 = \cdots = r_p = 2q$, and let G' be equal to G with the edge of K_2, first constituent graph, removed. Note that adding any edge to G' produces a graph whose blocks are a necklace graph and/or cliques. Given this, it is straightforward (and left to the reader) to verify that $rd(G')$ is achieved precisely when u, v are selected so that $G' + uv = G$, and that $\mathrm{rd}(G) = p/q$.

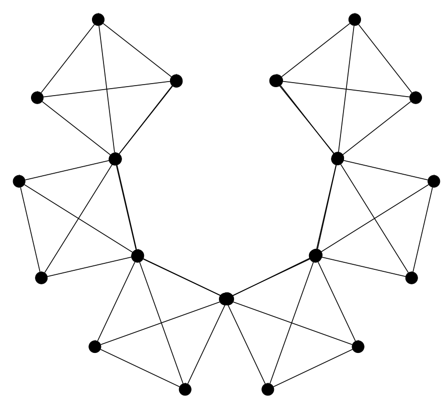

FIGURE 2. A graph G' with $p = 6$ and $q = 2$ and $rd(G') = p/q = 3$.

The graph G given is clearly claw-free, and clearly not planar. Thus for $\mathrm{rd}(G)$ we are currently in the same situation as with $\mathrm{dep}(G)$.

Question 2. Let p, q be positive integers, $p < q$. Is $\mathrm{rd}(G) = p/q$ constructible via planar graphs?

The question above is made more interesting by the fact that a large number of chemical graphs are planar.

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