Generalizing D-Graphs

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Abstract

Let $\omega_0(G)$ denote the number of odd components of a graph G. The deficiency of G is defined as $def(G) = \max_{X\subseteq V(G)}(\omega_0(G-X)-|X|)$, and this equals the number of vertices unmatched by any maximum matching of G. A subset $X \subseteq V(G)$ is called a Tutte set (or barrier set) of G if $def(G) = \omega_0(G-X)-|X|$, and an extreme set if $def(G-X) = def(G)+|X|$. Recently a graph operator, called the D-graph $D(G)$, was defined that has proven very useful in examining Tutte sets and extreme sets of graphs which contain a perfect matching. In this paper we give two natural and related generalizations of the D-graph operator to all simple graphs, both of which have analogues for many of the interesting and useful properties of the original.

KEYWORDS: Matching, D-graph, Tutte set, Barrier set, Extreme set

1 The D-graph of a graph with a perfect matching

A good reference for any terms left undefined is [3]. We consider a simple graph G. The *deficiency* of G is defined as

$$def(G) = \max_{X\subseteq V(G)}(\omega_0(G-X)-|X|)$$

where $\omega_0(G-X)$ denotes the number of odd components of $G-X$. Equivalently, and more intuitively, $def(G)$ can be shown to be the number of vertices of G unmatched by a maximum matching [4].

A *Tutte set* (also called a *barrier set* in [4]) of G is a subset $X \subseteq V(G)$ such that $\omega_0(G-X)-|X| = def(G)$. A set of vertices X in $V(G)$ is *extreme* if $def(G-X) = def(G)+|X|$. In [1, 2], a new graph operator was introduced to aid in the investigation of Tutte sets and extreme sets of graphs which had perfect matchings. Given a graph G with a perfect matching, the D-graph $D(G)$ is the graph whose vertex set and edge set are as follows:

1. $V(D(G)) = V(G)$, and
2. $xy \in E(D(G))$ if and only if $G - x - y$ has a perfect matching.

It is our goal in this paper to generalize the D-graph operator to facilitate analysis of Tutte and extreme sets in arbitrary graphs.

In [1, 2], another equivalent definition of $E(D(G))$ was given. We require the following notation. Let G be a graph, and let M denote some maximum matching of G. By $P_M[x, y]$, we denote an M-alternating path in G which joins vertices x and y, and which begins and ends with edges of M. In [4] it is shown that, if M is a perfect matching, then $G - x - y$ has a perfect matching if and only if $P_M[x, y]$ exists, and that the existence of such a path is independent of the choice of perfect matching M of G. We thus have the following equivalent definition of $xy \in E(D(G))$.

Proposition 1.1. *Let G be a graph with a perfect matching M. Then $xy \in E(D(G))$ if and only if there exists a path $P_M[x, y]$ in G.*

This alternating path characterization of the edges of $D(G)$ was used in [1] to examine the structure of D-graphs. In [2], these structural results were used to show that finding maximum Tutte sets is NP-hard for many classes of graphs (triangle-free, 2-connected planar, k-connected for any $k \geq 2$) and polynomial in several others (elementary graphs, 1-tough graphs). It was also shown that finding a maximal Tutte set can be accomplished in polynomial time.

The following four theorems summarize the main structural results from [1].

Theorem 1.2. *Let G be a graph with a perfect matching, and let $X \subseteq V(G)$. Then X is an extreme set of G if and only if X is an independent set in $D(G)$.*

Since maximal extreme sets are also maximal Tutte sets [1], we have the following equivalences.

Theorem 1.3. *Let G be a graph with a perfect matching and let $X \subset V(G)$. The following are equivalent:*

(i) X is a maximal Tutte set in G,

(ii) X is a maximal extreme set in G,

(iii) X is a maximal independent set in $D(G)$.

Theorems 1.2 and 1.3 provide a new means to investigate Tutte and extreme sets of graphs with perfect matchings.

Considered solely as a graph operator, the D-graph $D(G)$ was also shown to have interesting properties when iterated. First, the following was shown.

Theorem 1.4. *Let G be a graph with a perfect matching. Then $D(G)$ contains an isomorphic copy of G.*

Now let $D^k(G)$ denote the D-graph operator applied k times to the graph G, e.g. $D^2(G) = D(D(G))$ and so on. Since $V(G) = V(D(G))$, Theorem 1.4 implies that for some iteration we must have $D^{k+1}(G) \cong D^k(G)$, and that this stability continues for any future iterations. A surprising discovery was how quickly the D-graph operator converged, regardless of the structure of the original graph G.

Theorem 1.5. *Let G be a graph with a perfect matching. Then $D^3(G) \cong D^2(G)$.*

The minimum positive integer k such that $D^{k+1}(G) \cong D^k(G)$ is called the level of G and is denoted $level(G)$. Theorem 1.5 states that if G has a perfect matching, then $level(G) \leq 2$.

2 Generalized D-Graphs

The D-graph operator can be naturally generalized to all graphs in the following manner.

Definition 2.1. *Let G be a graph. We define $D(G)$ as follows:*

1. *$V(D(G)) = V(G)$, and*
2. *$xy \in E(D(G))$ if and only if $def(G - x - y) \leq def(G)$.*

Note that G has a perfect matching if and only if $def(G) = 0$, and thus the operator defined above is indeed a generalization of the operator of [1, 2]. Note also that under this definition the statement $xy \in E(D(G))$ is equivalent to the statement that if M and M' are maximum matchings of G and $G - x - y$ respectively, then $|M| \leq |M'| + 1$.

For graphs with $def(G) \geq 1$, the properties of this generalized D-graph operator closely parallel the properties shown in [1], as will be shown in this section. In order to examine the behavior of $D(G)$ we will utilize the Gallai-Edmonds decomposition of the graph G. Given a graph G, define the following sets of vertices:

$$A(G) = \{x \in V(G) \,|\, x \text{ is unmatched by some maximum matching of } G\},$$

$$B(G) = \{x \in V(G) - A(G) \,|\, x \text{ is adjacent to some vertex of } A(G)\},$$

$$C(G) = V(G) - A(G) - B(G).$$

When the graph G is apparent, we denote the Gallai-Edmonds decomposition of G by simply A, B and C. In the rest of this paper, A, B and C will typically indicate the decomposition of an arbitrary but fixed graph G upon which we will be using the D-graph operator, and the Gallai-Edmonds decompositions of the resulting graphs will be indicated with the parenthetical notation, e.g. the set of vertices of $D(G)$ which are unmatched by some maximum matching of $D(G)$ is $A(D(G))$, etc.

A *near-perfect* matching of G is a matching which leaves only one vertex unmatched, and a *factor-critical* graph is a graph in which $G-x$ has a perfect matching for all $x \in V(G)$. The following facts about $A(G), B(G)$, and $C(G)$ can be found in [4].

Theorem 2.2. *Let G be a graph and $A(G)$, $B(G)$, and $C(G)$ be the Gallai-Edmonds decomposition of G. Then:*

(a) the components of the subgraph induced by $A(G)$ are factor-critical;

(b) the subgraph induced by $C(G)$ has a perfect matching;

(c) if M is a maximum matching of G, then M consists of a near-perfect matching of each component of $A(G)$, a perfect matching of $C(G)$, and a matching of all vertices of $B(G)$ with vertices in distinct components of $A(G)$;

(d) B is a Tutte set of G; in particular, $def(G) = \omega_0(G[A]) - |B|$;

(e) a maximal Tutte set of G consists of B together with a maximal Tutte set of $G[C]$.

We will also need the following lemma from [4].

Lemma 2.3. (Stability Lemma) *Let G be a graph and A, B and C be the Gallai-Edmonds decomposition of G. Then:*

(a) if $x \in A(G)$, then $A(G-x) \subseteq A(G)-x$, $B(G-x) \subseteq B(G)-x$, and $C(G-x) \supseteq C(G)-x$.

(b) if $x \in B(G)$, then $A(G-x) = A(G)-x$, $B(G-x) = B(G)-x$ and $C(G-x) = C(G)-x$,

(c) if $x \in C(G)$, then $A(G-x) \supseteq A(G)-x$, $B(G-x) \supseteq B(G)-x$, and $C(G-x) \subseteq C(G)-x$,

The following lemma collects facts about edges of $E(D(G))$ that will prove useful. For notational purposes, it will be convenient to abbreviate $D(G) = D$, so for example the subgraph of $D(G)$ induced by $A(G)$ is indicated by $D[A]$.

Lemma 2.4. *Let G be a graph and A, B and C denote the Gallai-Edmonds decomposition of G.*

(a) If there is a maximum matching M such that $P_M[x,y]$ exists, then $xy \in E(D(G))$.

(b) If $x \in A(G)$, then $xy \in E(D(G))$ for all $y \in V(G)$.

(c) $D[B]$ is independent,

(d) there are no edges between $D[B]$ and $D[C]$,

(e) $D(G[C]) = D[C]$.

PROOF: (a) Let x' and y' denote the mates of x and y under M. Since $P_M[x,y]$ exists, in $G-x-y$ there exists a matching M' and a path $P_{M'}[x',y']$ consisting of $P_M[x,y]-x-y$ with the alternations reversed. Now simply note that $|M'| = |M|-1$, and thus $def(G-x-y) \leq def(G)$.

(b) Let $x, y \in A$, and so by definition there is some maximum matching M of G that misses x. But M is also a maximum matching of $G-x$, so if M' is a maximum matching of $G-x-y$, obviously $|M'| \geq |M|-1$.

Let $x \in A$ and $y \in B$, and let M be a maximum matching of G which misses x. By the Stability Lemma, part (b), $M' = M - y$ is a maximum matching of $G-y$. Thus $|M| = |M'|+1$, and so $def(G-y) = def(G)+1$. But since M misses x, then $def(G-x-y) = def(G-y)-1 = def(G)$.

Now let $x \in A$ and $y \in C$, and note that $def(G-y) = def(G)+1$. But by part (c) of the Stability Lemma $x \in A(G-y)$ and so $def(G-x-y) = def(G-y)-1 = def(G)$.

(c) If $x, y \in B$, and let M be a maximum matching of G. Then it follows from part (b) of the Stability Lemma that $M-x-y$ is a maximum matching of $G-x-y$. It only remains to note that $|M-x-y| = |M|-2$.

(d) Let $x \in B$, $y \in C$. By part (b) of the Stability Lemma it follows that $def(G-x) = def(G)+1$, and that $y \in C(G-x)$. Thus no maximum matching of $G-x$ misses y and $def(G-x-y) \geq def(G-x) \geq def(G)+1$.

(e) Let $xy \in E(D(G[C]))$. Since $G[C]$ has a perfect matching, this means that some $P_M[x,y]$ exists. But by (a), this means that $xy \in E(D[C])$. Thus $D(G[C]) \subseteq D[C]$.

Now let $xy \in E(D[C])$, and assume that $xy \notin E(D(G[C]))$. By definition, this means that $def(G-x-y) \leq def(G)$ and $def(G[C]-x-y) = def(G[C])+2 = 2$. Now let S be some Tutte set of $G[C]-x-y$, so that $def(G[C]-x-y) = \omega_0(G[C]-x-y-S)-|S|$. Now we have

$$\begin{aligned}
def(G-x-y) &\geq \omega_0(G-x-y-S-B)-|S|-|B| \\
&= \omega_0(G[C]-x-y-S)-|S|+\omega_0(G[A])-|B| \\
&= def(G[C]-x-y)+def(G) \\
&= 2+def(G) \\
&\geq 2+def(G-x-y),
\end{aligned}$$

which is a contradiction, meaning we must have $xy \in E(D(G[C]))$. Thus $D[C] \subseteq D(G[C])$, and so $D[C] = D(G[C])$ as claimed. $\square$

Part (a) of the previous lemma differs from Proposition 1.1 in that it gives only a sufficient condition for the existence of an edge of $D(G)$. Necessity also follows in the case that G has a perfect matching but not in general, a fact demonstrated by the following example. Let G be any graph which has a perfect matching. For any vertex v in G, let H denote the graph with vertex set $V(G) \cup \{x\} \cup \{y\}$ and edge set $E(G) \cup \{xv\} \cup \{yv\}$. It is straightforward to see that H does not have a perfect matching and that $xy \in E(D(H))$, but there is no maximum matching M of H such that $P_M[x,y]$ exists.

We now give the properties of the generalized D-graph operator analogous to Theorems 1.2 through 1.5. We consider first the iterative behavior of the operator.

Theorem 2.5. *Let G be a graph. Then $D(G)$ contains an isomorphic copy of G.*

Proof: We need only consider graphs which do not have a perfect matching. Let G be such a graph, let M be any maximum matching of G, and let $A_M \subseteq A$ be the vertices of G unmatched by M. Now observe the following:

(a) $D[G - A_M]$ contains an isomorphic copy of $D(G - A_M)$;

(b) $D[A]$ is a clique; and

(c) $D(G)$ contains the join of A and $B \cup C$.

The second and third follow directly from Lemma 2.4(b) so we consider statement (a). M is a perfect matching in $G - A_M$, so an edge $xy \in E(D(G - A_M))$ occurs if and only if a path $P_M[x, y]$ exists in $G - A_M$. However, by Lemma 2.4 this path guarantees the existence of $xy \in D(G)$, and thus $xy \in E(D[G - A_M])$. The theorem is now an immediate consequence of (a)-(c) and Theorem 1.4. $\square$

Theorem 2.6. *For any graph G on n vertices, $D^3(G) \cong D^2(G)$. In fact if G does not have a perfect matching, then $D^2(G) = K_n$.*

Proof: Again we need only consider the case where G does not have a perfect matching. Let M be a maximum matching of G and A_M be the set of vertices unmatched by M. As in the proof of Theorem 2.5, $D[G - A_M]$ has a perfect matching, $D[A]$ is a clique, and all possible edges exist between $D[A]$ and $D[B \cup C]$. We distinguish two cases:

Case 1: $|A_M| = 1$. Then M is a maximum matching of D. Let $x, y \in V(D)$. Now if $x \in A_M$ then by Lemma 2.4 we have $xy \in E(D^2(G))$. So we may assume that both $x, y \in V(D) - A_M$. Now $P_M[x, y] = xx'zy'y$ exists, where $A_M = \{z\}$ and x', y' are the mates of x, y respectively under M. Hence $xy \in E(D^2(G))$.

Case 2: $|A_M| > 1$. Then one maximum matching M' of D consists of M together with a maximum matching of the clique $D[A_M]$. Let $x, y \in V(D)$. If either x or y is in A_M, then Lemma 2.4 again gives $xy \in E(D^2(G))$. So we may assume that both $x, y \in V(G) - A_M$. Now $P_{M'}[x, y] = xx'zz'y'y$ exists, where $z \in A_M$ and x', y', z' are the mates of x, y, z under M' respectively. Hence $xy \in E(D^2(G))$.

In both cases $xy \in E(D^2(G))$, and so $D^2(G) = K_n$ as required. $\square$

Now we proceed to demonstrate that the $D(G)$ operator possesses properties concerning Tutte and extreme sets of general graphs similar to the properties it possesses in graphs with perfect matchings.

Theorem 2.7. *Let G be a graph and let $X \subseteq V(G)$ with $|X| > 1$. Then X is an extreme set of G if and only if X is an independent set of $D(G)$.*

Proof: ($\Rightarrow$) Let X be an extreme set in G, and thus $def(G-X)-|X| = def(G)$. Say vertices $a, b \in X$ are not independent in $D(G)$ and so by definition of $E(D(G))$ we have $def(G-a-b) \leq def(G)$. By Lemma 3.3.1 of [4], we have $def(G-S) \leq def(G)+|S|$ for any $S \subseteq V(G)$; in particular, $def(G-X) = def(G-a-b-(X-a-b)) \leq def(G-a-b)+|X|-2$. But now we have

$$\begin{aligned} def(G) &= def(G-X)-|X| \\ &\leq def(G-a-b)+|X|-2-|X| \\ &= def(G-a-b)-2 \\ &\leq def(G)-2, \end{aligned}$$

which is a contradiction.

($\Leftarrow$) If X' is an extreme set then it is easily verified [4] that any subset $X \subseteq X'$ is also an extreme set, and so we need only show the theorem true for a maximal independent set of $D(G)$. In [1] it is shown that any maximal extreme set is also a maximal Tutte set, so it suffices to show that a maximal independent set X of $D(G)$ is also a maximal Tutte set of G. Finally by Theorem 2.2(e), this is equivalent to showing that X is equal to B together with a maximal Tutte set of $G[C]$.

Let X be a maximal independent set of $D(G)$. Since $|X| > 1$ by Lemma 2.4 no vertex of A is an element of X, and so X is a maximal independent set of $D[B \cup C]$. By Lemma 2.4(c), B is a set of isolated vertices in $D[B \cup C]$. Thus X consists of B together with a maximal independent set of $D[C]$, and it only remains to show that a maximal independent set of $D[C]$ is a maximal Tutte set of $G[C]$. But by Lemma 2.4(e), $D(G[C]) = D[C]$ and any maximal independent set of $D[C]$ is also a maximal independent set of $D(G[C])$. Since $G[C]$ has a perfect matching, a maximal independent set of $D[C]$ is also a maximal Tutte set of $G[C]$, and the proof is complete. $\square$

Corollary 2.8. *Let G be a graph and let X be a subset of $V(G)$ with $|X| > 1$. The following are equivalent:*

(i) X is a maximal Tutte set in G,

(ii) X is a maximal extreme set in G,

(iii) X is a maximal independent set in $D(G)$.

The result above is the best possible for general graphs in the following sense. The condition that all singleton vertices are extreme (or Tutte) characterizes those graphs which have a perfect matching [5]. Thus, if a graph does not have a perfect matching then we are guaranteed that some set X with $|X| = 1$ is neither extreme nor Tutte. Hence, the $|X| > 1$ condition is necessary when considering arbitrary graphs.

3 D_*-Graphs

In this section, we introduce a second generalization of the original D-graph operator on graphs with perfect matchings.

Definition 3.1. *Let G be a graph. We define the D_*-graph $D_*(G)$ as follows:*

1. *$V(D_*(G)) = V(G)$, and*
2. *$(x, y) \in E(D_*(G))$ if and only if $def(G - x - y) = def(G)$.*

We note two immediate facts. First, when G does not possess a perfect matching, then $D_*(G)$ is a subgraph of $D(G)$. Second, note that under this definition $xy \in E(D(G))$ is equivalent to saying that if M and M' are maximum matchings in G and $G - x - y$ respectively, then $|M| = |M'| + 1$.

Despite the relationships just mentioned between $D_*(G)$ and $D(G)$, the behavior of the D_*-graph operator is in many ways markedly different than that of the D-graph operator. For example, under the D_*-graph operator the existence of an M-alternating path $P_M[x, y]$ no longer guarantees an edge $xy \in E(D_*(G))$. The alternating path condition is valid only for the smallest deficiencies.

Theorem 3.2. *Let G be a graph with $def(G) \leq 1$. If there is a maximum matching M such that $P_M[x, y]$ exists, then $xy \in E(D_*(G))$.*

Proof: We need only consider the case $def(G) = 1$. Let $x, y \in V(G)$ be such that $P_M[x, y]$ exists. Note that $|V(G)|$ is odd, and thus $def(G-x-y) \geq 1$. After reversing the alternations of the path $P_M[x, y]$, we obtain a matching M' of $G-x-y$ of cardinality $|M'| = |M|-1$, and so $def(G-x-y) \leq def(G) = 1$. Thus $def(G - x - y) = def(G) = 1$. □

To see that the condition $def(G) \leq 1$ is necessary, take any graph H with a perfect matching M, let $ab \in M$, and form G by appending two pendant vertices each to both a and b. Denoting one pendant vertex of a by x and one pendant vertex of b by y, we see that a maximum matching of G is $M' = M-ab+ax+by$, and thus that $P_{M'}[x, y]$ exists. However $def(G) = 2$ and $def(G-x-y) = 0$, and $xy \notin E(D_*(G))$. The example can accommodate graphs of higher deficiency by simply appending additional pendant edges to either a or b.

Another difference between $D_*(G)$ and $D(G)$ is that it is no longer the case that $D_*(G)$ must contain an isomorphic copy of G. Consider $G = H + I$, the join of any nonempty, n-vertex graph H to an independent set I of size at least $n + 2$. In $D_*(G)$ the edges of H disappear, and only the edges between H and I remain, so $D_*(G)$ is isomorphic to $K_{n,|I|}$, a proper subgraph of G.

Despite these differences, there is still a relationship between the Tutte sets and extreme sets of G and the independent sets of $D_*(G)$. In order to set out this relationship, we make the following observations.

Lemma 3.3. *Let G be a graph and let A, B and C denote the Gallai-Edmonds decomposition of G. Then*

(a) each component of $D_*[A]$ is complete,

(b) there is a complete bipartite join from $D_*[A]$ to $D_*[B \cup C]$,

(c) $D_*[B]$ is independent,

(d) there are no edges between $D_*[B]$ and $D_*[C]$,

(e) $D(G[C]) = D_*[C]$,

Parts (c) and (d) follow immediately from the fact that they hold for $D(G)$ as well, and $D_*(G) \subseteq D(G)$. The proofs of parts (b) and (e) are almost identical to the proofs of parts (b) and (e) of Lemma 2.4 and so are omitted. All that remains is to prove (a).

PROOF OF (A): Assume otherwise. Then $D_*[A]$ contains an induced path of length three. Let xyz be such a path, and observe that this implies that $def(G-x-y) = def(G-y-z) = def(G)$. Thus, any maximum matching of G that includes y must leave both x and z unmatched. Let M be such a matching. Similarly, every maximum matching that includes x leaves y unmatched. Let M' be a matching that includes x. If M' does not include z, then $def(G-x-z) = def(G)$ and $xz \in E(D_*(G))$. Otherwise, consider the symmetric difference $M \Delta M'$. It is well known that the components of the symmetric difference of any two maximum matchings is the union of even alternating paths and even alternating cycles (see for instance [6]), and so x and z must be end vertices of distinct alternating paths in $M \Delta M'$. Let P_x be the alternating path of $M \Delta M'$ that ends at x, and let $M_x = M \Delta P_x$. Clearly, this is a maximum matching which includes x but not z, and hence $def(G-x-z) = def(G)$. So $xz \in E(D_*(G))$, contradicting our choice of x, y, z. $\square$

We now have the following.

Theorem 3.4. *Let G be a graph and let $X \subseteq V(G)$ be an independent set of $D_*(G)$ with $|X| > 1$. Then either*

(1) X is an extreme set of G, or

(2) $X \subseteq A$.

PROOF: Comparing Lemmas 3.3 and 2.4 reveals that $D[B \cup C] = D_*[B \cup C]$, and a complete bipartite join exists between $D_*[B \cup C]$ and $D_*[A]$. Thus $D(G)$ and $D_*(G)$ differ only in the edges present within A, and the result follows. $\square$

Corollary 3.5. *Let G be a graph and let $X \subseteq V(G)$ be a maximal independent set of $D_*(G)$ with $|X| > 1$. Then either*

(1) X is a maximal extreme set and a maximal Tutte set of G, or

(2) $X \subseteq A$.

4 Iterated D_*-Graphs

Given a graph G, we say that $level_*(G) = i$ if i is the smallest nonnegative integer such that $D_*^{i+1}(G)$ is isomorphic to $D_*^i(G)$. While Theorem 2.6 shows that $level(G) \leq 2$ for all G, in this section we show that $level_*(G) \leq 4$ and characterize the level$_*$ 4 graphs in terms of their Gallai-Edmonds decomposition.

To this end we examine the possible maximum matchings of $D_*(G)$ by considering the Gallai-Edmonds decomposition of G and Lemma 3.3. As with the $D(G)$ operator, for notational simplicity we often abbreviate $D_* = D_*(G)$.

In the case where $\omega_0(D_*[A]) \leq |B| + |C|$, the odd components of $D_*[A]$ may be matched near perfectly, leaving $\omega_0(D_*[A])$ unmatched vertices among them. We will match these vertices with vertices in $B \cup C$ until we have exhausted the remaining vertices from the odd components of $D_*[A]$. In the event that $|B| > \omega_0(D_*[A])$, we will now match the remaining vertices of B to even components of $D_*[A]$ in pairs. In either case, considering that $D_*[C]$ has a perfect matching and the even components of $D_*[A]$ are complete, we leave at most one vertex of $B \cup C$ unmatched. Hence, in this case, $def(D_*(G)) \leq 1$.

We now show that $\omega_0(D_*[A]) \leq |B| + |C|$ implies that $level_*(G) \leq 3$.

Theorem 4.1. *Let G be a graph such that the Gallai-Edmonds decomposition of G obeys $\omega_0(D_*[A]) \leq |B| + |C|$. Then $level_*(G) \leq 3$. Furthermore, if $\omega_0(D_*[A]) < |B| + |C|$ then $D_*^2(G)$ is complete.*

Proof: If $def(D_*(G)) = 0$, then $D_*(G)$ has a perfect matching and from previous results we know that $level_*(D_*) \leq 2$ and thus $level_*(G) \leq 3$. Identical reasoning allows us to assume that $\omega_0(D_*[A]) < |B| + |C|$. We therefore examine the structure of $D_*^2(G)$ when $\omega_0(D_*[A]) < |B| + |C|$ and $def(D_*(G)) = 1$.

Under the $D(G)$ operator every $x \in A$ was adjacent to every $y \in V(G)$, a circumstance which is only true for the smallest deficiencies under the D_* operator. The following proposition is immediate from the definition of A.

Proposition 4.2. *Let G be a graph with $def(G) = 1$. If $x \in A$, then $xy \in D_*(G)$ for all $y \in V(G)$.*

In the maximum matching of $D_*(G)$ described previously, the lone unmatched vertex belonged to $B \cup C$. Since $D_*(G)$ contains the join of A and $B \cup C$, it is easy to see that we may take this unmatched vertex to be any vertex of $B \cup C$, and so $B \cup C \subseteq A(D_*(G))$. Since $def(D_*(G)) = 1$ we have that $D_*^2[B \cup C]$ is complete and that D_*^2 contains the join of A and $B \cup C$. We need only determine what edges exist in $D_*^2[A]$.

Let e denote a vertex in an even component of $D_*[A]$. Since there is an unmatched vertex $x \in B \cup C$ under M, the matching $M' = M - ee' + e'x$ is also a maximum matching, and so $e \in A(D_*(G))$. Thus by Proposition 4.2 each of these vertices is joined to every vertex in $D_*^2(G)$, and we need only consider which edges exist between the odd components of $D_*[A]$ in $D_*^2(G)$.

Let a_1 and a_2 denote two vertices in odd components of $D_*[A]$. If a_1 and a_2 are in the same component of $D_*[A]$ they are joined in $D_*^2(G)$, so assume that they lie in different components. If $|C| > 0$, then there exists at least one edge, say between $c_1, c_2 \in C$, that is not a part of a maximum matching of $D_*(G)$. We may assume that in a maximum matching M of $D_*(G)$, the mates of a_1, a_2 are $a_1' = c_1$ and $a_2' = c_2$ respectively, and thus the M-alternating path $a_1c_1c_2a_2$ path exists, and by Theorem 3.2 we have $a_1a_2 \in E(D_*^2(G))$.

Now assume $|C| = 0$. If there are any vertices e, e' of an even component of $D_*[A]$, then the M-alternating path $a_1a_1'ee'a_2'a_2$ exists. Furthermore, if any odd components have more than three vertices, then the path $a_1a_1'dd'a_2'a_2$ exists, where d, d' are two vertices of that odd component. In both circumstances the edge $a_1a_2 \in E(D_*^2(G))$. Finally if $|C| = 0$ and the only components of $D_*[A]$ are singletons, then $|A| = \omega_0(D_*[A]) < |B|$, an impossibility if the original graph G did not have a perfect matching.

We see that if G satisfies $\omega_0(D_*[A]) \leq |B| + |C|$ then either $D_*(G)$ has a perfect matching or else $D_*^2(G)$ is complete. In particular, $level_*(G) \leq 3$. □

Now we examine the case when $\omega_0(D_*[A]) > |B|+|C|$. In this case, as $D_*(G)$ contains the join of A and $B \cup C$, we can match one vertex in each odd component of $D_*[A]$ with a vertex in $B \cup C$, leaving $def(D_*(G)) = \omega_0(D_*[A]) - |B| - |C| > 0$ vertices unmatched. We will now investigate the level$_*$ of graphs of this type. The case when $def(D_*(G)) = 1$ is handled separately.

Theorem 4.3. *When $\omega_0(D_*[A]) > |B| + |C|$ and $def(D_*(G)) = 1$, then $D_*^3(G) = K_n$ and $level_*(G) = 3$.*

Proof: Recall that $A(D_*(G)) = F$, $B(D_*(G)) = B \cup C$, and $C(D_*(G)) = E$ where E and F denote the vertices of the even and odd components of $D_*[A]$ respectively, and also that the unmatched vertices belong to the odd components of $D_*[A]$. By Lemma 3.3(b-e) we have that a bipartite join exists between F and $B \cup C \cup E$ in $D_*^2(G)$, that $B \cup C$ is independent in $D_*^2(G)$, that no edges exist between $B \cup C$ and E, and that $D_*^2[E] = D_*[E]$. It only remains to establish what edges exist within $D_*^2[F]$. However, by Proposition 4.2 from the previous case if $x \in D_*[F] = F = A(D_*(G))$ then x is connected to every other vertex in $D_*^2(G)$.

The structure of $D_*^2(G)$ is now determined: the independent vertices of $B \cup C$ and the complete even components $D_*[E]$ are joined to K_f, where $f = |F|$. We examine three maximum matchings of interest. One maximum matching, call it M_1, induces a perfect matching in E, matches $B \cup C$ to F, and finally takes a near-perfect matching in the remaining portion of the K_f. One unmatched vertex remains in F, and thus $F \subseteq A(D_*^2(G))$. Alternatively, we may form a second maximum matching M_2 by matching this unmatched vertex with a vertex of F, leaving a vertex of $B \cup C$ unmatched. Thus $B \cup C \subseteq A(D_*^2(G))$. Alternatively, working again with M_1, we can match the unmatched vertex of F with a vertex of E, leaving a vertex of E unmatched. This matching M_3 shows that $E \subseteq A(D_*^2(G))$. We conclude that $A(D_*^2(G)) = V(G)$, and so $D_*^3(G) = K_n$. □

Theorem 4.4. *Let G be a graph with $def(D_*(G)) \geq 2$. Then $level_*(G) \leq 4$, with equality if and only if one of the following two conditions holds:*

(1) $|B|+|C|+|E|+1=\omega_0(D_*[A])$*, or*

(2) $|B|+|C|+|E|=\omega_0(D_*[A])$ *and* $\omega_0(D_*[A])<|F|$.

Proof: The only cases remaining occur when $\omega_0(D_*[A]) > |B|+|C|$ and $def(D_*(G)) \geq 2$. Since $A(D_*(G)), B(D_*(G))$, and $C(D_*(G))$ are unchanged from the $def(D_*(G)) = 1$ case, everything follows exactly as in the previous proof except for the determination of what edges exist in $D_*^2[F]$. Obviously if $x, y \in F$ are in the same component of $D_*[F]$, then they are adjacent in $D_*^2(G)$. But if $x \in F_x$, $y \in F_y$ where F_x, F_y are distinct components of $D_*[F]$, then since $def(D_*(G)) \geq 2$ and a complete bipartite join exists in $D_*(G)$ between $A \supseteq F$ and B by Lemma 3.3, it is easy to see that a maximum matching exists that misses both x and y. Hence $D_*^2[F] = D_*[F]$.

The analysis of $D_*^3(G)$ now breaks down into the following cases.

Case I: $\omega_0(D_*^2[F]) \geq |B|+|C|+|E|+2$. Here a maximum matching in $D_*^2(G)$ matches each of the vertices of $B \cup C \cup E$ to a distinct component of $D_*^2[F] = D_*[F]$ together with a maximum matching in the remaining vertices of $A(D_*^2(G)) = F$. Thus $A(D_*^2(G)) = F$ and $B(D_*^2(G)) = B \cup C \cup E$. Using Lemma 3.3 we see that $B \cup C \cup E$ is independent in $D_*^3(G)$ and a complete bipartite join exists between $B \cup C \cup E$ and F. Within F, analysis similar to that done earlier in this proof shows that $D_*^3[F] = D_*^2[F] = D_*[F]$. In $D_*^3(G)$ we see again that $A(D_*^2(G)) = F$ and $B(D_*^2(G)) = B \cup C \cup E$, and that $D_*^4[F] = D_*^3[F]$. Thus $D_*^4(G) = D_*^3(G)$ and $level_*(G) \leq 3$.

Case II: $\omega_0(D_*^2[F]) = |B|+|C|+|E|+1$. A maximum matching in $D_*^2(G)$ is formed exactly as in the previous case, only now $def(D_*^2(G)) = 1$. In this case, we note that $level_*(D_*(G)) = 3$ by Theorem 4.3, and hence $level_*(G) = 4$.

Case III: $\omega_0(D_*^2[F]) = |B|+|C|+|E|$. The maximum matching described in the two previous cases is here a perfect matching, and so by Theorem 1.5, we have $level(D_*^2(G)) = level_*(D_*^2(G)) \leq 2$, and so $level_*(G) \leq 4$. Additionally, we note that when $|F| = \omega_0(D_*[F]) = f$, then $f = \frac{n}{2}$, and $D_*^4(G) = D_*^3(G) = K_{f,f}$ and $level_*(G) \leq 3$. Otherwise, we have $|F| = f > \frac{n}{2}$, and in this case $D_*^3(G) = K_f \wedge \overline{K_{n-f}}$ and $D_*^4(G) = K_n$ and $level_*(G) = 4$.

Case IV: $\omega_0(D_*^2[F]) < |B|+|C|+|E| = |B(D_*(G))| + |C(D_*(G))|$. Then by Theorem 4.1, we have $D_*^2(D_*(G))$ is complete and so $level_*(D_*(G)) \leq 2$ and $level_*(G) \leq 3$.

The theorem statement is obtained by observing that by definition $\omega_0(D_*[A]) = \omega_0(D_*[F])$ and that in all cases $D_*[F] = D_*^2[F]$. $\square$

Finally we provide examples of each type of level$_*$ 4 graphs. Let T_j denote a triangle with $3j+4$ pendant vertices appended to one vertex. Then $T_j \cup jK_{1,2}$ is an example of the first type, while $T_j \cup jK_{1,2} \cup K_{2,3}$ is an example of the second type.

5 Open Problems

Two natural problems come to mind in regard to D-graphs and D$_*$-graphs. First, is there a natural characterization of D-graphs? In other words, when is a graph H a D-graph or D$_*$-graph, i.e. when is $H = D(G)$ or $H = D_*(G)$ for some G? At the moment it appears difficult to determine even what form such a characterization would take. For example we mention that no characterization is possible in terms of forbidden subgraphs. To see this take any proposed forbidden subgraph H and create the graph H' by appending to each vertex $v \in V(G)$ a new pendant vertex v'. The graph H' has a perfect matching and it is not difficult to see that $D(H') = D_*(H') = H'$. In particular, the original graph H is an induced subgraph of $D(H')$.

Finally, now that D-graphs and D$_*$-graphs are defined for any graph G, is it possible to characterize the D-graphs or D$_*$-graphs of various special classes of graphs? By Theorems 2.7 and 3.4 and Corollaries 2.8 and 3.5, such a characterization would yield information about the Tutte sets and extreme sets of those graph classes.

The authors would like to thank the anonymous referees for many helpful comments which improved the exposition of this paper, and in particular for providing the argument used in the proof of Lemma 3.3(a).

References

[1] D. Bauer, H. J. Broersma, A. Morgana, and E. Schmeichel. Tutte sets in graphs I: maximal Tutte sets and D-graphs. *J. Graph Theory*, to appear.

[2] D. Bauer, H. J. Broersma, N. Kahl, A. Morgana, E. Schmeichel, and T. Surowiec. Tutte sets in graphs II: the complexity of finding maximum Tutte sets. *Disc. Appl. Math.*, to appear.

[3] B. Bollobás, Modern Graph Theory, Springer-Verlag, New York, 1998.

[4] L. Lovász and M. D. Plummer. Matching Theory. *Ann. Discrete Math. Vol. 29*, North-Holland Publishing Co., Amsterdam, 1986.

[5] S. Wang and J. Hao. The extreme set condition of a graph. *Disc. Math.*, 260:151 − 161, 2003.

[6] D. West, Introduction to Graph Theory, Prentice Hall, Inc., Saddle River, NJ, 1996.