Dr. Charles L. Suffel: Scholar, Teacher, Mentor, Friend

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Abstract

Dr. Charles L. Suffel (1941-2021) was an infuential mathematics educator and scholar at Stevens Institute of Technology for more than half a century. The Managing Editor of *Networks* for 20 years, Suffel's reach extended far beyond the Stevens campus. He coauthored dozens of graph theory papers and mentored more than a dozen Ph.D. theses. In this article, we discuss his contributions to the field of network reliability theory and his legacy as a teacher and mentor.

1 Introduction

Our friend and mentor, Charlie Suffel, passed away in February of 2021. While no single article can possibly do him justice, we will attempt to here. We represent the research group that Charlie Suffel maintained for most of his time as a graph theorist; the group began at Stevens Institute of Technology in Hoboken, New Jersey, where Suffel taught for more than 50 years. Suffel was the Managing Editor of *Networks* from 1979 to 1999, and he and the late Frank Boesch (Editor-in-Chief of *Networks* for more than 20 years) began a weekly Wednesday night seminar, which began at 9 PM after graduate courses were completed, so all the graph theorists at Stevens (housed variously in the mathematics, electrical engineering, computer science and operations research programs) could view presentations in the field by faculty, research students or visitors. A significant component of these meetings, in keeping with Suffel and Boesch's down-to-earth personalities, was the choice of refreshment: pizza and beer.

After the seminar ended in the 1990s, the group met on a semi-regular basis (sans pizza and beer) and included faculty from both Stevens and Seton Hall as well as Stevens graduate students and Seton Hall undergraduates. Suffel took special pride in these sessions and always offered constructive suggestions. Numerous papers and theses resulted from these meetings, and a listing is included later in this article.

2 Suffel's Research

Charlie Suffel received his Ph.D. from Brooklyn Polytechnical University in 1969 under the direction of the functional analyst George Bachmann. He also worked for a stint at Bell Labs before joining Stevens Institute of Technology as an assistant professor. The first half dozen publications of his academic career were in the area of functional analysis and topological vector spaces. Promoted to full professor in 1979, Suffel would go on to co-author more than more than 50 papers in the areas of graph theory and network reliability theory. The first of these studied subgraphs of Eulerian graphs [15]; coauthored with Boesch and Ralph Tindell, it appeared in the first edition of *Journal of Graph Theory*. In fact, his most frequent collaborators were Boesch (27 co-authored papers), Dan Gross (23), John T. Saccoman (13) and Bill Kazmierczak (12), the latter two being among Suffel's 15 completed Ph.D. students. What should be noted is that Suffel continued to be an active researcher during his 20 years (1995-2015) as the Dean of Graduate Studies at Stevens; in fact, 30 of his papers were published during that time, and more than half of his Ph.D. students graduated in that period. His research efforts were recognized by Stevens when he was honored as a co-recipient of the Jess H. Davis Memorial Award for Research Excellence.

2.1 All-terminal Reliability and Spanning Trees

Graph vulnerability parameters, those parameters of graphs that in some sense measure the 'strength' of a graph, were of lifelong interest to Suffel, particularly questions of extremality. Let $\mathcal{G}_{n,m}$ denote the class of simple graphs with n nodes and m edges. Important examples of extremality questions are: Which graphs in particular $\mathcal{G}_{n,m}$ classes are the strongest or the weakest (by some measure)? Do these strongest or weakest graphs have easily identifiable features in general? Are there particular operations that will uniformly make a graph stronger or make it weaker? Suffel was able to make significant contributions to all of these questions, and for a wide variety of such parameters.

One graph vulnerability parameter which has garnered particular attention and generated an extensive literature is the all-terminal reliability of G, denoted $R_G(p)$, which is the probability that G is still connected if each edge of G fails independently with probability p. In general $R_G(p)$ is $\#P$-hard to compute, with the difficultly scaling exponentially in the number of nodes and edges [48]. In [11] however Suffel et al. were the first to prove that for some $\mathcal{G}_{n,m}$ there existed a *uniformly optimally reliable graph*, a graph $G \in \mathcal{G}_{n,m}$ for which $R_G(p) \geq R_H(p)$ for any $p \in (0,1)$ and any $H \in \mathcal{G}_{n,m}$. Uniformly optimally reliable graphs, the 'strongest' graphs by the all-terminal reliability measure, have been discovered for a number of other graph classes [1, 24, 38, 46, 47, 49, 50], which include a few by Suffel students and frequent co-authors Dan Gross and John T. Saccoman.

The coefficients of the polynomial $R_G(p)$ contain combinatorial information about G. The coefficient of the lowest order term of $R_G(p)$ is $\tau(G)$, the number of spanning trees of G, sometimes called the *complexity* of G. The complexity $\tau(G)$ is itself a widely-recognized measure of graph and network strength, and here too Suffel made important contributions to extremality problems. Uniformly optimally reliable graphs are necessarily τ-optimal, i.e., have the most spanning trees in their $\mathcal{G}_{n,m}$ class. Beyond the uniformly optimally reliable graphs already mentioned however, Suffel et al. [42] identified τ-optimal graphs in additional $\mathcal{G}_{n,m}$ classes, and he was one of the first to look at the problem in the broader context of multigraphs [25]. A number of others, including his student Louis Petingi, have identified more types of τ-optimal graphs in various classes [21, 35, 36, 37, 43, 44]. The coefficient of the highest order term of $R_G(p)$ is called the reliability domination of G and is denoted $d(G)$. Suffel was among the first to investigate $d(G)$, helping to demonstrate its relationships to important combinatorial parameters such as acyclic orientations, Whitney's broken cycles, and Tutte's internal activity associated with the chromatic polynomial [12]. He also identified extremal graphs for $d(G)$ [13]. For details and other results we refer the reader to the survey [14], which was co-authored by Suffel.

Some of the tools used in the previous results deserve special mention. Let $N_G(u)$ and $N_G(v)$ respectively denote the open neighborhoods in G of u and v. The *compression of G from u to v* produces a new graph H by, for each $x \in N_G(u) - N_G(v) - \{v\}$, removing all edges from G of the form ux and replacing them with corresponding edges of the form vx. An inverse operation, call it decompression, can be defined if a graph G contains a node v which *dominates* another node u, that is, if they satisfy $N_H(u) - \{v\} \subseteq N_H(v) - \{u\}$. In this case, for any collection of $x \in N_H(v) - N_H(u) - \{u\}$, we produce a new graph G by removing edges of the form vx and replacing them with corresponding edges of the form ux. An illustration of these operations appears in Figure 1.

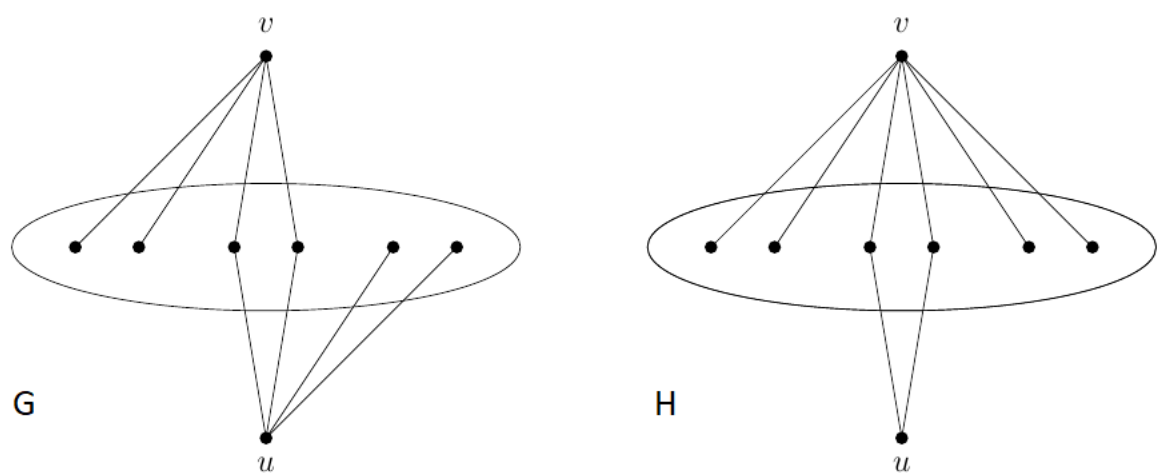

Figure 1: An illustration of graph compression and decompression. The compression of G from u to v transforms the graph G on the left to the graph H on the right. Decompression on H from v to u transforms the graph on the right to the graph on the left. Note that decompression requires node v to dominate node u.

Graph decompression appears to have been first employed by Kelmans in [36] who showed that, when applicable, it increases all-terminal reliability for all $p \in (0, 1)$, and can be used to identify uniformly most reliable graphs in the most edge-dense $\mathcal{G}_{n,m}$ classes. Suffel et al. independently rediscovered decompression in [47] (calling it 'the swing surgery'), and reproved Kelmans's reliability results. But in [47] Suffel et al. were also the first to recognize that the compression operation had important extremal consequences as well. They showed that repeated application of graph compression can transform any $G \in \mathcal{G}_{n,m}$ into a *threshold graph*, a well-known and much-studied class of graphs (see for example [40]). Since compression uniformly decreases all-terminal reliability, this establishes that for any $G \in \mathcal{G}_{n,m}$ there is a threshold $H \in \mathcal{G}_{n,m}$ such that $R_H(p) \leq R_G(p)$, and thus minimum graphs for all-terminal reliability must necessarily be threshold.

The result of Suffel et al. [47] on threshold graphs and graph compression has had important consequences for a wide variety of extremality problems: if a graph parameter can be shown to be decreased (resp. increased) by graph compression, then necessarily threshold graphs minimize (resp. maximize) that graph parameter. In this way threshold graphs have been shown to be extremal for a surprising variety of graph parameters, and in the rest of this section we hope to give the reader some idea of their scope. Threshold graphs have been shown to minimize other graph vulnerability parameters such as number of spanning trees [16], toughness, edge toughness, and binding number [32], and they maximize scattering number and rupture degree [31, 32]. Graph compression has also been shown to affect a number of other, non-vulnerability graph parameters as well. Threshold graphs minimize the number of k-factors for any k [23], minimize the number of k-matchings for any k [18], maximize the largest root of the matching polynomial [18], maximize both the number of independent sets of order r and the number of cliques of order r for any r [18], minimize the smallest real root of the independence polynomial [18], minimize the magnitude of the i^{th} coefficient of the chromatic polynomial for any i [45], minimize the magnitude of the i^{th} coefficient of the Laplacian polynomial for any i [18], maximize the spectral radius [17], and maximize the number of homomorphisms into certain target graphs [19, 20]. Most recently, a far-reaching generalization of Suffel's original results on all-terminal reliability has shown that a vast array of parameters associated with the Tutte polynomial are decreased, or in rare cases increased, by graph compression [33]. (The all-terminal reliability polynomial $R_G(p)$ is a particular evaluation of the Tutte polynomial of G.) Parameters decreased include the number of spanning forests and the number of spanning connected subgraphs; the number of acyclic orientations, totally cyclic orientations, acyclic orientations with a single source, or score vectors of orientations; enumerations of a wide variety of different types partial orientations; the magnitude of the coefficients of the flow polynomial; and the number of critical configurations of level i of the Abelian sandpile model. By the result of Suffel et al. in [47], then, all of the parameters listed are minimized by threshold graphs. In addition, in [33] evaluations of the q-state Potts model from theoretical physics for $q \geq 1$ are shown to be increased by compression, hence these are maximized by threshold graphs.

2.2 Component Order Reliability

While giving a talk on the residual node reliability model at the Clemson Mini-Conference on Discrete Mathematics and Algorithms, Frank Boesch was told by a member of the audience that he could not study this model.

Traditionally, the reliability of a graph was the all-terminal reliability. This model was based on the telephone network, where the phones are the nodes, and the wires are the edges. The old phones were virtually indestructible, the network was only interrupted when the phone lines went down. With the introduction of computers and computer networks the scenario had changed. The computers often crashed, while the links remained operational. A different model was needed.

Let $G = (V, E)$ be a graph with node set V and edge set E and let $n = |V|$ and $m = |E|$. In this model edges are perfectly reliable, but nodes operate with equal but independent probability p. A subset $W \subseteq V$ is an operating state if the subgraph induced by W is connected. The residual node reliability, $R_n(G, p)$, is the probability that the surviving nodes form an operating state. The assumption that the nodes operate independently of each other allows us to express $R_n(G, p)$ in the following form: $R_n(G,p) = \sum_{i=1}^{n} S_i(G)p^i(1-p)^{n-1}$, where $S_i(G)$ is the number of operating states of order i.

The complaint was because the model was not *coherent*. A collection of subsets of a set is coherent if whenever a subset belongs to the collection, then every superset of it also belongs. Consider, for example, the graph K_{100} with a pendant edge uv between the node u on the K_{100} and a node v disjoint from the K_{100}. If nodes u and v fail, the surviving nodes induce a K_{99}, thus is an operating state. However, if only u fails, the surviving nodes induce a disconnected graph, so is not an operating state. Thus, the collection of operating states is not coherent; a superset of an operating state need not be operational. Another drawback of this model is that relatively small operating states are tolerated. If in the example only u and v are operating then an operating state exists, even though 99% of the nodes failed.

While Boesch and Suffel did not appreciate being told what they could study, they did acknowledge that there were some problems with the model. On the drive home they tried to see if they could come up with a model that addressed these problems. The scenario they came up with was the following. Consider a distributive computer network in which, upon the failure of some of the computers, the network may still be operational, even if disconnected, as long as there remains a "sufficiently large" portion of the network which is still interconnected. Now if additional computers came back online the network would remain operational. Also, the specification of what is "sufficiently large" would preclude small operating states. Suffel, Boesch, and Gross (BGS), began to work on developing a new reliability model based on this new notion.

Let $G = (V, E)$ be a graph with node set V and edge set E and let $n = |V|$ and $m = |E|$. Let k be a fixed integer, with $0 < k \leq n$. In this model edges are perfectly reliable, but nodes operate with equal but independent probability p. A subset $W \subseteq V$ is an operating state if the subgraph induced by W contains a connected component of order at least k. The *k node operating component reliability of* G, denoted $R_\alpha^{(k)}(G,p)$, is the probability that the surviving nodes form an operating state, i.e., it contains a component of order at least k. The assumption that the nodes operate independently of each other allows us to express $R_\alpha^{(k)}(G,p)$ in the following form: $R_\alpha^{(k)}(G,p) = \sum_{i=k}^{n} A_i^{(k)}(G)p^i(1-o)^{n-1}$, where $A_i^{(k)}(G)$ is the number of operating states of order i, i.e., an i-node induced subgraph of G that contains a connected component of order at least k.

In [6, 7, 8] BGS introduced $R_\alpha^{(k)}(G,p)$ and studied several properties of it. Since any i-node induced subgraph of G that contains a connected component of order at least $k+1$ contains a connected component of order at least k, it follows that $A_i^{(k+1)}(G) \leq A_i^{(k)}(G)$ for every i, thus $R_\alpha^{(k+1)}(G,p) \leq R_\alpha^{(k)}(G,p)$. Examples were given that show the computation of $R_\alpha^{(k)}(G,p)$ is more involved than just deleting the first term of $R_\alpha^{(k+1)}(G,p)$. It is known that the calculation of $R_n(G,p)$ is #P-hard. This implies that the calculation of $R_\alpha^{(k)}(G,p)$ for all k has to be #P-hard. If for each k, $R_\alpha^{(k)}(G,p)$ could be computed in polynomial time, then for each k, $A_k^{(k)}(G)$ could be computed in polynomial time. Since $A_k^{(k)}(G) = S_k(G)$, the computation of $R_n(G,p)$ could be done in polynomial time. Since the collection of operating states is coherent, the Kruskal-Katona inequality was used to find bounds for the reliability.

2.3 Component Order Connectivity

Boesch, Gross, and Suffel turned their attention to studying the vulnerability of a network. For the residual node reliability model, the network is not operational or has failed if the deletion of nodes results in a disconnected graph. The minimum number of nodes whose deletion disconnects the graph is the (node) connectivity and is denoted by κ. In [9, 10] BGS introduced and began to study the analogous vulnerability parameter for the component order connectivity model. Recall in this model that nodes fail and edges do not. Given a predetermined value k, $2 \leq k \leq n$, a set of nodes W is an operating state if it induces a subgraph that contains a component of order at least k; otherwise we say W is a failure state. The k component connectivity or simply component order connectivity, denoted $\kappa_c^{(k)}$, is the minimum $|U|$ such that U is a subset of V and all components of the subgraph induced by $V - U$ have order $\leq k - 1$, i.e., $V - U$ is a maximum order failure state.

If the failure of a set of nodes produces a surviving subgraph with no component of order at least k, then there are no components of order at least $k+1$. Thus, it follows that $\kappa_c^{(n)} \leq \kappa_c^{(n-1)} \leq \cdots \leq \kappa_c^{(2)}$. Some of these values are easy to calculate: $\kappa_c^{(n)} = 1$ for any graph; $\kappa_c^{(n-1)} = 1$ if there is a cutpoint, 2 otherwise. On the other end of the spectrum, the calculation of $\kappa_c^{(2)}$ is #P-hard for an arbitrary graph, as $\kappa_c^{(2)}$ of a graph is equal to n minus the independence number of the graph.

Failure states can be formed two different ways: either the subgraph induced by $V - U$ is disconnected and all components have order $\leq k-1$, or the induced subgraph is connected and has order $\leq k-1$. In the former case $|U| \geq \kappa$ while the latter case $|U| \geq n-k+1$. But if $|U| \geq n-k+1$, then $V - U$ is always a failure state, thus $\kappa_c^{(k)} \leq n - k + 1$. These conditions are used to establish the relationship between κ and $\kappa_c^{(k)}$, namely that if $\kappa \geq n - k + 1$ then $\kappa_c^{(k)} = n - k + 1$, and if $\kappa \leq n - k$ then $\kappa \leq \kappa_c^{(k)} \leq n - k$.

A graph G in $\mathcal{G}_{n,m}$ is max κ if $\kappa(G) \geq \kappa(H)$ for all H in $\mathcal{G}_{n,m}$. Similarly, a graph G in $\mathcal{G}_{n,m}$ is max $\kappa_c^{(k)}$ if $\kappa_c^{(k)}(G) \geq \kappa_c^{(k)}(H)$ for all H in $\mathcal{G}_{n,m}$. If δ is the minimum degree of a graph, then $\left\lfloor \frac{2m}{n} \right\rfloor \geq \delta \geq \kappa$ and a graph G is max κ if and only if $\left\lfloor \frac{2m}{n} \right\rfloor = \delta = \kappa$. If $\left\lfloor \frac{2m}{n} \right\rfloor \geq n - k + 1$, then every max κ graph is max $\kappa_c^{(k)}$, with $\kappa_c^{(k)} = n - k + 1$; if $\left\lfloor \frac{2m}{n} \right\rfloor = n - k$, then every max κ graph is max $\kappa_c^{(k)}$, with $\kappa_c^{(k)} = n - k$. For the case $\left\lfloor \frac{2m}{n} \right\rfloor < n - k$ there exists classes of graphs where it is impossible to simultaneous maximize κ and $\kappa_c^{(k)}$.

At this point, thesis students were brought into the research group to work on the models, the first being L. William Kazmierczak. In [3] they found a condition that guarantees a strong network design for relatively dense graphs. The main result follows: If $\delta = \left\lfloor \frac{2m}{n} \right\rfloor \geq \left\lfloor \frac{n}{2} \right\rfloor$ and C_4 is not contained in the complement of G, then G is max κ and G is max $\kappa_c^{(k)}$, for $k \geq 3$. These graphs also have diameter ≤ 2. In [3] constructions were also provided of graphs satisfying these conditions. In [34] they asked the following questions: (1) Given $a \geq n - k + 1$, does there exist a graph G on n nodes with $\kappa = a$ and $\kappa_c^{(k)} = n - k + 1$, and (2) Given $a \leq b \leq n - k$, does there exist a graph G on n nodes with $\kappa = a$ and $\kappa_c^{(k)} = b$. They were able to answer both questions in the affirmative.

2.4 Component Order Edge Connectivity

The research group then turned their attention to an analogous vulnerability parameter for edge failure. In the traditional edge-failure model the network is operational if, after the failure of edges, the remaining edges induce a connected graph; in this case we say the remaining edges then constitute an operating state. The vulnerability parameter associated with this mode is the edge connectivity λ, which is the minimum number of edges whose deletion disconnects the graph. For this model, the collection of operating states is coherent, so the deficiencies associate with residual node connectivity are no longer present. Still, as with the previously discussed component order connectivity model, there may be situations for which it is not necessary that the surviving subgraph is connected, but only that there is a connected component of order at least some predetermined threshold value. The group was joined at this point by A. Suhartomo, a student of Boesch. In [4] they introduced the parameter *component order edge connectivity*.

Given $2 \le k \le n$, the *k-component edge connectivity*, or simply the component order edge connectivity of G, denoted by $\lambda_c^{(k)}$, is the minimum number of edges whose deletion results in a subgraph with all components of order $\le k-1$. Unlike the case for $\kappa_c^{(k)}$, a failure state for this model must disconnect the graph, so $\lambda \le \lambda_c^{(k)}$, for each k, $2 \le k \le n$. In fact, $\lambda \le \lambda_c^{(n)} \le \lambda_c^{(n-1)} \le \cdots \le \lambda_c^{(2)} = m$. They showed for any graph G and any value of k, $\kappa_c^{(k)} \le \lambda_c^{(k)}$, which is analogous to the well-known result that $\kappa \le \lambda$. Is it possible to simultaneously maximize both $\kappa_c^{(k)}$ and $\lambda_c^{(k)}$ for all k? The answer is no, as can be shown even for trees. Consider the collection of trees on n nodes, and let P_n be the path, $K_{1,n-1}$ be the star, and T_n be any other tree. Then it is not difficult to show that $\kappa_c^{(k)}(K_{1,n-1}) \le \kappa_c^{(k)}(T_n) \le \kappa_c^{(k)}(P_n)$ and $\lambda_c^{(k)}(P_n) \le \lambda_c^{(k)}(T_n) \le \lambda_c^{(k)}(K_{1,n-1})$. Thus, for any value of k, the best tree for $\kappa_c^{(k)}$ is the worst tree for $\lambda_c^{(k)}$, while the best tree for $\lambda_c^{(k)}$ is the worst tree for $\kappa_c^{(k)}$.

By this time the group was joined by John T. Saccoman, who had been a student of Suffel. In [2] they showed the following result: Let G be a connected group of order n. If $\delta(G) \ge \left\lfloor \frac{n}{a+1} \right\rfloor$, $1 < a \le k-1$, then $\lambda_c^{\left\lceil \frac{n}{a} \right\rceil}(G) \ge \delta(G)$. This generalizes the result of Chartrand, that if $\delta(G) \ge \left\lfloor \frac{n}{2} \right\rfloor$, then $\lambda(G) = \delta(G)$, since when $a = 1$, $\lambda_c^{(k)} = \lambda$. Unfortunately, shortly after the completion of this paper our colleague Frank Boesch passed away.

2.5 Weighted Component Edge Connectivity

The group, which now had Lakshmi Chandra as a member, next considered the notion of weighted component edge connectivity, a generalization of component order edge connectivity [29, 30]. In this model positive weights $w(v)$ are assigned to each node v. Let $W = \sum_{v \in V} w(v)$ and $\bar{w} = \max\{w(v) | v \in V\}$. Given a predetermined threshold value k, with $\bar{w} \le k \le W$, the weighted component edge connectivity of G, denoted by $\lambda_{wc}^{(k)}$, is the minimum number of edges whose deletion results in a subgraph with all components having total weight $\le k-1$.

If $w(v) = \alpha$ for each node, then $\lambda_{wc}^{(k)} = \lambda_c^{\left\lceil \frac{k}{\alpha} \right\rceil}$, in particular when $\alpha = 1$ we have $\lambda_{wc}^{(k)} = \lambda_c^{(k)}$. In [29, 30] the following problem were considered: Given $k - 1 \ge w_1 \ge w_2 \ge \cdots \ge w_n > 0$, determine trees T_M and T_m on n nodes and an assignment of the weights to the nodes such that $\lambda_{wc}^{(k)}(T_M)$ is maximum over all trees on n nodes and $\lambda_{wc}^{(k)}(T_m)$ is minimum over all trees on n nodes. In the uniformly weighted case the problems are satisfied by $K_{1,n-1}$ and P_n, respectively. For the general weighted case, the maximum value is achieved by assigning w_i to the center node of $K_{1,n-1}$. The minimum value is achieved by applying a bin packing algorithm to the nodes of P_n, where the bins are consecutive nodes.

Efficient algorithms to compute $\lambda_{wc}^{(k)}$ of a tree and of a unicycle were also provided. It should be noted such algorithms were provided for all the component order parameters introduced.

2.6 Neighbor-Component Order Connectivity

In [26, 27] Gunther and Hartnell first introduced the idea of *neighbor-connectivity*. The notion of neighbor-connectivity provides important information on how reliable a network can be when failures of a node may impact its neighbors. With the neighbor-connectivity parameter, the failure of a node causes the deletion of its closed neighborhood, i.e., the node and its adjacent neighbors as well. The minimum number of closed neighborhoods whose removal results in an empty, complete, or disconnected subgraph is called the neighbor-connectivity of the graph.

Considered as a reliability model, the state of the associated system consists of those nodes which are operating, i.e., those nodes that have neither failed nor are in the closed neighborhood of a failed node. Note that with this definition, small disconnected components also survive. This is not what one may expect in the previous component order models. Instead, we would expect that large components are still operating, but not the small ones. Adapting neighbor-connectivity to a component order model, Kristi Luttrell, a student of Kazmierczak, introduced the vulnerability parameter *neighbor-component order connectivity* [39]. Given $1 \leq k \leq n$, the k-neighbor-component order connectivity of G, denoted by $\kappa_{nc}^{(k)}(G)$ or simply $\kappa_{nc}^{(k)}$ when G is understood, is the minimum $|D|$ such that $D \subseteq V$ and $\langle V - N[D] \rangle$ is a failure state, i.e., all components of $\langle V - N[D] \rangle$ have order $\leq k-1$.

One immediate result is that for any graph G on n nodes, $\kappa_{nc}^{(k)}(G) \leq \kappa_c^{(k)}(G)$ for $2 \leq k \leq n$. An arbitrary tree provides interesting realizability results pertaining to the aforementioned parameters. It has been previously shown that the component order connectivity of an arbitrary tree of order n lies between that of the star graph and the path graph of order n [9, 10]. However, the neighbor-component order connectivity of a tree can be larger than that of the path. Since the component order connectivity of a graph is at least as large as the neighbor-component order connectivity, then an upper bound for the neighbor-component order connectivity of an arbitrary tree is the component order connectivity of the path [22]. For any tree T_n on n nodes, $1 = \kappa_c^{(k)}(K_{1,n-1}) \leq \kappa_c^{(k)}(T_n) \leq \kappa_c^{(k)}(P_n) = \left\lfloor \frac{n}{k} \right\rfloor$.

We observe that with a threshold value of 1 the neighbor-component order connectivity of a graph is equivalent to the domination number of the graph. Given a graph G, a node v in the graph is said to *dominate* itself and all of its neighbors, i.e., v dominates the nodes in the closed neighborhood $N[v]$. A set S of nodes in G is a *dominating set* of G if every node in G is dominated by at least one node in S, i.e., every node not in S is adjacent to at least one node in S. The *domination number*, denoted by $\gamma(G)$ or simply γ, is defined to be the minimum cardinality among the dominating sets of nodes in G. Since $\kappa_{nc}^{(1)}(G) = \gamma(G)$, the problem of computing the neighbor-component order connectivity of an arbitrary graph for an arbitrary threshold value is #P-hard.

A graph parameter ρ has the subgraph property if whenever H is a subgraph of G, $\rho(H) \leq \rho(G)$. Unlike $\kappa_c^{(k)}$ which has the property, $\kappa_{nc}^{(k)}$ does not.

2.7 Neighbor-Component Order Edge Connectivity

The final vulnerability parameter is the edge version of neighbor-component order connectivity. The parameter, introduced by Monica Heinig in [28], is neighbor-component edge connectivity. In this model edges may fail, and when an edge fails its end nodes are subverted, thus all other edges emanating from the end-nodes are rendered inoperable and are removed from the graph. Alternatively, when an edge fails, the resulting subgraph is the graph that results from the deletion of the end-nodes.

Let $2 \leq k \leq n$. Upon the failure of some edges, and the removal if any other edges incident on the edges, the resulting subgraph is in a failure state if all components have order $\leq k-1$. The neighbor-component order edge connectivity, denoted $\lambda_{nc}^{(k)}$, is the minimum number of edge failures needed to create a failure state.

One immediate result is for any graph G on n nodes, $\lambda_{nc}^{(k)}(G) \leq \lambda_{c}^{(k)}(G)$. Since the failure of an edge deletes its end nodes, $\kappa_{c}^{(k)}(G) \leq 2\lambda_{nc}^{(k)}(G)$. An important result that can be used to calculate the value of $\lambda_{nc}^{(k)}$ is that there exists a minimum failure set of edges that is a matching. A final result gives an upper bound on the value of $\lambda_{nc}^{(k)}$, namely $\lambda_{nc}^{(k)} \leq \lceil \frac{n-k+1}{2} \rceil$.

3 Suffel's Teaching and Mentorship

Dr. Charles Suffel taught for more than 50 years at Stevens Institute of Technology. A mathematician by trade, he also taught courses in electrical engineering, computer science and operations research. He was known for his humor, rigor, and attention to detail. Suffel received numerous awards for the quality of his teaching, including the Stevens Alumni Association Outstanding Teacher Award in 1978 and 1991 and the Henry Morton Distinguished Teaching Professor Award in 1989. It is rare to find a professor who excels in both the teaching aspects of the profession and the research side. While he received several research grants over his academic career, Suffel's main contribution to the field was his mentorship of Ph.D. theses.

3.1 Ph.D. Theses

Dr. Charles Suffel was a very well-respected thesis advisor. His ability to bring students on board and prepare them for the rigors of Ph.D. research is legendary at Stevens and in the Graph Theory community. In an unironic fashion, he referred to his Ph.D. students as his academic "children" and displayed as much pride in their activities, frankly, as would any biological parent. In fact, the word "family" pops up in most reminiscences about him, and you will see that in the subsequent sections of this article. One of the co-authors of this article, Dr. Kristi Luttrell, was a Ph.D. student of Dr. L. William Kazmierczak, and, of course, was referred to by Suffel as his "granddaughter." Not surprisingly, "Grandpa" was a member of Luttrell's thesis committee.

The following table lists all of the Ph.D. students of Charlie Suffel. All his students' Ph.D.s are from Stevens Institute of Technology. If those students mentored Ph.D. students, their names are listed under "Descendents." Those descendents who earned their degrees from Stevens are indicated with an asterisk.

3.2 Stevens Institute of Technology Teaching

The material in this section is taken from a tribute to Suffel that is a part of the Stevens Institute of Technology website [51].

Alicia Muth, doctoral candidate in the Department of Mathematical Sciences, came to consider Suffel family as he mentored her over the years. "In my first semester at Stevens," she recalled, "I was falling behind in his advanced calculus course. When I spoke to him about the difficulties I

Table 1: Charlie Suffel academic geneology

Name	**Year**	**Descendants**
John Burns	1976	
Douglas Bauer	1978	Lewis Lasser*, Linda McGuire*, Michael Yatauro*, Aori Nevo*
Lynne Doty	1986	
Laura Schoppmann	1990	
Constantine Stivaros	1990	
Louis Petingi	1991	Mohammed Talafha
Zhong Chen	1994	
John T. Saccoman	1995	
Jay Stiles	1998	
L. William Kazmierczak	2003	Kristi Luttrell*
Nathan Kahl	2005	
Nikolay Strigul	2007	
James Weatherall	2009	Benjamin Feintzeig, Samuel Fletcher, Marian Gilton, Sarita Rosenstock
Lakhmi Vidyasagar	2013	

was having, rather than dismiss me, he understood and began meeting with me weekly." Muth and Suffel went on to collaborate on a paper about graph theory, published and presented at the Stevens Graduate Research Conference, MAA MathFest, and the Southeastern International Conference on Combinatorics, Graph Theory, and Computing.

Linda Habermann-Ward, assistant in the Department of Mathematical Sciences, said that Suffel was her first supervisor when she began working at Stevens more than 20 years ago. According to Haberman-Ward, Suffel was a "tough but fair" instructor. "His students built personal relationships with him that lasted decades and honestly, they loved and adored him and he them," she said. "He was so positive and encouraging. That carried over into everything he did. He always had a kind word and a smile. You couldn't help but feel happiness when you saw him walk in."

Robert Gilman, professor of mathematics, recalled, "I remember that Charlie used to joke a lot about how tough he was on students. He did expect a lot from his students, but he also cared a lot about them. If a student was having difficulty but working hard, Charlie was always there to help."

4 Personal Reminscenses

One former student, Stevens graduate Brian King, was quoted in the Stevens alumni magazine The Indicator regarding Suffel, "I'm certain that, in most instances, he was the smartest person in the room, but he never portrayed himself that way."

4.1 Fr. Gabriel Costa, Ph.D.

Faith was a major component in Charlie Suffel's life. His dear friend and confidante, Fr. Gabriel Costa, Ph.D., celebrated Charlie Suffel's funeral mass. Here is his reflection on his friend.

Actual conversation between two Stevens Tech Ph.D. students, circa 1971:

Graduate Student 1: "I'm really glad that I decided to study math here at Stevens Tech."

Graduate Student 2: "Yeah, the math faculty is great. Brilliant... and nobody stands on ceremony."

Graduate Student 1: "You're right about that. Take Charlie Suffel. Great guy. After only two days, he's your friend."

Graduate Student 2: "It took you TWO days to become his friend!"

Charlie Suffel was the big brother I never had.

I met Charlie as a young Masters Degree student in 1970. We became fast friends, and I would often speak with him about Mathematics, Baseball, and - at times - issues about life. Often, I would seek his counsel... and I would never leave his office without something to ponder.

In 1972 I completed my masters degree. I decided to move on from Stevens, though not quite sure what the future would hold. At the age of twenty-four I was trying to discern; would it be Marriage, Priesthood, Graduate School, Industry, etc.?

Much would happen during the next eight years, including my becoming a priest and deciding to pursue doctoral studies in Mathematics.

So, I returned to my roots... my alma mater... Stevens Institute of Technology.

It was good to be back. And it was good to see Charlie again. I had put on some weight and Charlie was losing more hair (I couldn't resist). Charlie and I took up our friendship where we left off. And, over the ensuing years, I would be privileged to minister to Charlie and his family as a priest.

Oh... those were the days! Our families would come together there would be humor, celebrations, stories told over and over... feasts... true joy and true fellowship.

There would be nights of Poker and trips to Yankee Stadium. Families and friends would often break bread. There would be weddings and baptisms. There would be jokes and cigars with perpetual viewings of The Godfather. There would be visits to local pubs and the sketching of nuanced mathematical diagrams on paper napkins, while sitting at the bar, imbibing potent potables and wolfing down hamburgers.

And there would be support when friends and family members would be called Home to Eternity.

For the last twenty years or so, Charlie and I would chat on the phone once or twice a week. The topics were varied, and discussions could go on for over an hour. But the underlying message was always one of concern, support and brotherly love.

More than anything, I miss those calls.

Charlie left three legacies: His devotion to God and a commitment to live a Christian life... his unwavering love for and loyalty to his family and his friends... his passion for all things Mathematical (research and teaching).

While many feel that Charlie left this life much too soon, in truth, he epitomized the very reason why Jesus Christ came among us:

"I came that they might have life and have it more abundantly." - John 10:10

Professor Charlie Suffel lived life to the full.

4.2 Dr. Monika Heinig

Dr. Monika Heinig is the Director of Data Science and Analytics at the New York Company Clyde. Monika was a Ph.D. student of Suffel, his student in several graduate classes, and his teaching assistant. She says, “He saw the potential in people and wanted them to live up to that and would try to help in any way he could. He would push you because he knew you could do it, while providing guidance and support. And once you were one of his Ph.D. students, you became family; you were one of his academic children. And that made you want to be better because you wanted to make him proud.”

Regarding his teaching style, Dr. Heinig said, “He was the best lecturer I’ve personally ever seen. ... While he seemed to have completely memorized everything, he always prepped before every class and just spoke ‘in the order of the progression of things’ as he would explain it... He was an extremely fair and transparent professor. He would crack jokes and tell stories, and would always have a beer after class ‘but never before!’ ”

Of Charlie’s presence on the Stevens campus, Monika says, “He was such a staple both at the Math Department and at Stevens. I’ve only known both with him there and I can’t imagine either without him. Everyone seemed to know him and love him; He was kind of like a celebrity around campus. Anytime I interacted with anyone outside the math department and said I was Charlie’s Ph.D. student I would get the same reaction every time — a huge smile and ‘Oh Charlie? I know Charlie! He’s great!’ ”

Speaking for many of his Ph.D. students, Monika says that she will miss “his laugh, his stories that he would tell over and over again (without realizing he had already told it, probably several times), his generosity, his kindness, his wisdom, and his presence.” Echoing the sentiments of Brian King, Dr. Heinig says, “He could hold a conversation with anyone he met, usually about any topic. And as brilliant as he was, he never made anyone feel less than.”

The authors of this article concur. We miss him greatly.

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