Graph Vulnerability Parameters, Compression, and Quasi-Threshold Graphs

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Abstract

Kelmans, and later independently Bogdanowicz and Satyanarayana, Schoppmann, and Suffel, showed that a graph operation which has come to be known as the compression of G from vertex u to vertex v could not increase, and typically decreased, both the number of spanning trees and the all-terminal reliability of a graph. Both these quantities are well-known vulnerability parameters, i.e., measures of the strength of a network, and subsequently a number of other prominent vulnerability parameters—including vertex connectivity, toughness, scattering number, edge connectivity, edge toughness, and binding number—have been shown to be affected by compression in a similar way. As a consequence threshold graphs are extremal for all of the parameters mentioned. In this paper we show that for the graph vulnerability parameters integrity, tenacity, and k-component order connectivity, if u, v are adjacent then compression cannot increase, and typically decreases them. As a consequence, these parameters have quasi-threshold graphs as extremal graphs. We also show, however, that there are graphs with non-adjacent u, v where compression increases these parameters. To the best of our knowledge, these parameters are the first identified that behave differently under compression depending upon which pairs of vertices are used in the compression.

1 Introduction

Let G be a graph and let $u, v \in V(G)$, and let $N_G(u)$ and $N_G(v)$ denote the neighborhoods in G of u and v respectively. The *compression of G from u to v* produces a new graph $G_{u\to v}$ by, for each $x \in N_G(u) - N_G(v) - \{v\}$, removing all edges from G of the form ux and replacing them with corresponding edges of the form vx. An illustration of this operation appears in Figure 1.

The compression operation has also been called a 'path property transformation' [6], a 'shift transformation' [8, 10], a 'Kelmans transformation' [13, 14], and a 'swing surgery' [25]. It's earliest applications were in network reliability: Kelmans first showed that $\tau(G_{u\to v}) \leq \tau(G)$ and $Rel_{G_{u\to v}}(p) \leq Rel_G(p)$, where $\tau(G)$ and $Rel_G(p)$ represent respectively the number of spanning trees and the all-terminal reliability polynomial of G, and subsequently Bogdanowicz [6] rediscovered the spanning tree result and Satyanaraya, Schoppmann, and Suffel [25] rediscovered both the spanning tree and reliability results. Later Brown, Colbourn, and Devitt [10] showed that the transformation actually decreases every coefficient of the all-terminal reliability polynomial, generalizing both the spanning tree and reliability results. (The number of spanning trees is one of the coefficients of the all-terminal reliability polynomial.)

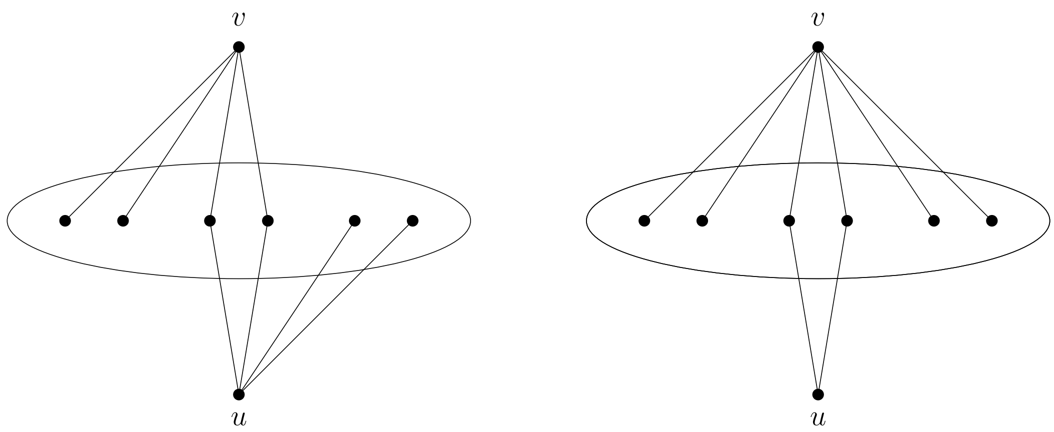

Figure 1: An illustration of compression. On the left is a graph G, and on the right $G_{u\to v}$, the compression of G from u to v. If edge uv had been present in G, it would also be present in $G_{u\to v}$.

Compression has since been shown to affect a number of other graph parameters as well. In what follows we will indicate by "decreases parameter p" the relationship $p(G_{u\to v}) \leq p(G)$, and in a similar manner use "increases." Compression has been shown to decrease the number of k-factors for every k [18], decrease the number of k-matchings for any k [21], increase the spectral radius of both G and its complement [14], increase the largest root of the matching polynomial [13], decrease the smallest real root of the independence polynomial and the coefficients of the chromatic polynomial [13], and increase the number of independent sets of order k for any k [13, 15]. And in [15, 16], whose "compression" terminology and $G_{u\to v}$ notation we follow, the result on number of independent sets was significantly generalized when compression from u to v was shown to increase the number of homomorphisms into certain target image graphs. Most recently, in a companion paper to this one [19] we examined a number of vulnerability parameters—vertex connectivity, toughness, scattering number, edge connectivity, edge toughness, and binding number—and showed that compression affects them in the same way that it affects $\tau(G)$ and $Rel_G(p)$ (which are also common vulnerability parameters). In all of these results, including the original ones on spanning trees and all-terminal reliability, compression acts upon a graph G in a uniform manner: given a parameter p, we have either $p(G) \geq p(G_{u\to v})$ or $p(G) \leq p(G_{u\to v})$, regardless of choice of G or $u, v \in V(G)$.

In this paper we examine another set of common vulnerability parameters—integrity, tenacity, k-component order connectivity, and rupture degree—and show that provided that u, v are adjacent we again have uniformity: $p(G_{u\to v}) \leq p(G)$ for p equal to integrity, tenacity, and k-component order connectivity for any k, and $r(G_{u\to v}) \geq r(G)$ for rupture degree. We are able to show in fact that rupture degree behaves uniformly, and $r(G_{u\to v}) \geq r(G)$ for any choice of G and $u, v \in V(G)$. However we also give an infinite family of graphs that show that integrity, tenacity, and k-component order connectivity can increase under compression when the distance between u and v is two. (When the distance between u and v is three or more $G_{u\to v}$ is disconnected, and while our results still hold here we are less interested in that case. For example, a number of graph vulnerability parameters, including number of spanning trees and reliability, are trivially zero for disconnected graphs.) To the best of our knowledge then, integrity, tenacity, and k-component order connectivity are the first parameters to exhibit this "non-uniform" behavior under compression.

We say *vertex v dominates vertex u* if $N_G(u) \subseteq N_G[v]$, where $N[v] = N_G(v) \cup \{v\}$. If $G_{u\to v} \neq G$ then the compression operation takes a graph where u, v do not dominate each other and produces a graph in which one does dominate the other. A *threshold graph* is a graph in which, given any pair of vertices, one must dominate the other, and a *quasi-threshold graph* is one in which, given any pair of adjacdent vertices, one must dominate the other. Bogdanowicz [6] and Satyanarayana, Schoppmann, and Suffel [25] also demonstrated that given any graph G, it is possible to produce a threshold graph H from G via a sequence of compression operations, and Cutler and Radcliffe [15, 16] demonstrated that it is possible to produce a quasi-threshold graph via a sequence of compression operations on adjacent vertices. As a consequence, in this paper we also obtain that threshold graphs maximize rupture degree, and quasi-threshold graphs minimize integrity, tenacity, and k-component order connectivity.

All graphs in this paper are assumed to be simple (although see [10] for an extension of compression to multigraphs). The vulnerability parameters considered, as well as the terms and notation necessary for those definitions, will be defined in the paper as they appear. For any undefined terms we refer the reader to a standard reference like [9]. Finally, if $N_G(u) - N_G(v) - \{v\}$ is empty then $G_{u\to v} = G$, and if $N_G(v) - N_G(u) - \{u\}$ is empty then $G_{u\to v}$ is isomorphic to G, with the isomorphism obtained by simply switching the labels of u and v. In these instances we clearly have the results indicated, so in the rest of the paper we may assume that $N_G(u) - N_G(v) - \{v\}$ and $N_G(v) - N_G(u) - \{u\}$ are both non-empty.

2 Compression with adjacent vertices

The vulnerability measures considered are all concerned with what happens when vertices are deleted from a graph. We begin then with a few simple but useful observations on vertex deletions and compression which for reference we give in a lemma.

Lemma 2.1. *Let G be a graph and $u, v \in V(G)$, with $X \subset V(G)$, and let C be a component of $G - X$.*

1. *If $u, v \notin V(C)$, then C is also a component of $G_{u\to v} - X$ on the same vertex set.*
2. *If $u, v \in V(C)$, then $G - X$ and $G_{u\to v} - X$ are identical except that C has been replaced by $C_{u\to v}$.*

Proof. To prove the first statement we show that the edges within C are identical in $G - X$ and $G_{u\to v} - X$, and then that there are no edges connecting any vertex of C to any non-C vertex in $G_{u\to v}$. The only edges that change endpoints during compression from u to v are edges that change an endpoint from u to v. Since $u, v \notin V(C)$, therefore no edges within C can have changed endpoints during compression, and thus the edges within C are identical in $G - X$ and $G_{u\to v} - X$. Now assume the contrary that there is an $x \in V(C)$ and $y \notin V(C)$ such that xy is an edge of $G_{u\to v} - X$. Since $xy \notin G - X$, then we must have either that xy was deleted, i.e. $y \in X$, or that xy was moved during compression, i.e. $v = y$. In the former case clearly xy is also deleted in $G_{u\to v}$, a contradiction. And in the latter case if vx is an edge of $G_{u\to v} - X$ that was moved then this implies ux was an edge of $G - X$. Then in $G - X$ we had $u \in V(C)$, a contradiction as well.

To prove the second statement we simply note that, since we are deleting vertices, $x \in N_{G-X}(u) - N_{G-X}(v) - v$ implies $x \in N_G(u) - N_G(v) - v$ and $x \notin N_{G-X}(u) - N_{G-X}(v) - v$ implies $x \notin$

$N_G(u) - N_G(v) - v$. Thus more generally when $u, v \notin X$ we have $(G - X)_{u\to v} = G_{u\to v} - X$, which implies that $u, v \in V(C)$ for a component C in $G - X$ means that $C_{u\to v}$ has replaced C. $\square$

We note that $C_{u\to v}$ in the above lemma is not necessarily a single component. If for instance $d_G(u, v) \geq 3$ then it is easy to see that u is an isolated vertex in $G_{u\to v}$, and therefore an isolated vertex in $C_{u\to v}$ as well.

Let $\omega(G)$ denote the number of components of a graph G and $c(G)$ denote the order of the largest component of G. We now give what is in many respects the main theorem of the section, as results on integrity, tenacity, and k-component order connectivity all follow from it.

Theorem 2.2. *Let G be a connected graph with $u, v \in V(G)$ such that $uv \in E(G)$. Then for every $X \subseteq V(G)$ there exists an $X' \subseteq V(G_{u\to v})$ such that all three of the following hold:*

1. $|X'| = |X|$,
2. $\omega(G_{u\to v} - X') \geq \omega(G - X)$,
3. $c(G_{u\to v} - X') \leq c(G - X)$.

Proof. We consider the following three cases. In the first two cases we show that we may take $X' = X$, in other words that $\omega(G_{u\to v} - X) \geq \omega(G - X)$ and $c(G_{u\to v} - X) \leq c(G - X)$. In the third and final case a simple alteration of X produces the requisite X'.

Case 1. $v \in X$. In this case note that $G_{u\to v} - v$ is the subgraph of $G - v$ created by the additional deletion of edges of the form ux for $x \in N_G(u) - N_G(v)$. Thus, letting $X = X'$, we have $G_{u\to v} - X' \subseteq G - X$, which implies both that $\omega(G_{u\to v} - X') \geq \omega(G - X)$ and $c(G_{u\to v} - X') \leq c(G - X)$.

Case 2. $u, v \notin X$. Since $uv \in E(G)$, in this case we must have u, v in the same component of $G - X$. Call this component C. Then, again letting $X' = X$, by Lemma 2.1 the components of $G - X$ are identical to the components of $G_{u\to v} - X'$ with the exception that C has been replaced by $C_{u\to v}$. But since $uv \in E(G)$ then $uv \in E(G_{u\to v})$ and $C_{u\to v}$ is, like C, a single component. This implies both $\omega(G_{u\to v} - X') = \omega(G - X)$ and $c(G_{u\to v} - X') = c(G - X)$.

Case 3. $u \in X$ and $v \notin X$. Let $X' = X - \{u\} \cup \{v\}$. Then $G_{u\to v} - X'$ is isomorphic to a subgraph of $G - X$; to see this note that $G_{u\to v} - v$ is isomorphic to a subgraph of $G - u$ by simply changing the label u to the label v in $G_{u\to v} - v$. But $G_{u\to v} - X'$ isomorphic to a subgraph of $G - X$ implies both $\omega(G - X) \leq \omega(G_{u\to v} - X')$ and $c(G_{u\to v} - X') \leq c(G - X)$. Finally, for this X' since $u \in X', v \notin X'$ we have $|X'| = |X - \{u\} \cup \{v\}| = |X|$. $\square$

The integrity of a graph G, denoted $I(G)$, was introduced by Barefoot, Entriger, and Swart in [2]. For a survey of structural results, bounds, and relationships between integrity and other graph parameters we refer to [1]. Graph integrity is defined as

$$I(G) = \min \{ |X| + c(G - X) \mid X \subseteq V(G) \} .$$

We call $X \subseteq V(G)$ an I-set of G if $I(G) = |X| + c(G - X)$. We now have the following.

Theorem 2.3. *Let G be a graph with $u, v \in V(G)$ such that $uv \in E(G)$. Then $I(G_{u\to v}) \leq I(G)$.*

Proof. Let X be an I-set of G, so that $I(G) = |X| + c(G - X)$. By Theorem 2.2 there exists a set X' such that $|X'| = |X|$ and $c(G_{u\to v} - X') \leq c(G - X)$. But this implies

$$I(G_{u\to v}) \leq |X'| + c(G_{u\to v} - X') \leq |X| + c(G - X) = I(G)$$

as required. □

The tenacity of a graph G, denoted $T(G)$, was introduced by Cozzens, Moazzami, and Stueckle in [11, 12], and for a survey of results we refer to [24]. Graph tenacity is defined as

$$T(G) = \min \left\{ \frac{|X| + c(G - X)}{\omega(G - X)} \;\middle|\; X \subset V(G) \right\}.$$

We call $X \subset V(G)$ a T-set of G if $T(G) = (|X| + c(G - X))/\omega(G - X)$. We now have the following.

Theorem 2.4. *Let G be a graph with $u, v \in V(G)$ such that $uv \in E(G)$. Then $T(G_{u\to v}) \leq T(G)$.*

Proof. Let X be a T-set of G, so that $T(G) = (|X| + c(G - X))/\omega(G - X)$. By Theorem 2.2 there exists a set X' such that $|X'| = |X|$ and $c(G_{u\to v} - X') \leq c(G - X)$ and $\omega(G - X) \leq \omega(G_{u\to v} - X')$. This implies

$$T(G_{u\to v}) \leq \frac{|X'| + c(G_{u\to v} - X')}{\omega(G_{u\to v} - X')} \leq \frac{|X| + c(G - X)}{\omega(G - X)} = T(G)$$

as required. □

The concept of component order connectivity was introduced in [3, 4] as generalization of the well-known connectivity parameter κ. The k-component order connectivity of a graph G, denoted $\kappa_c^{(k)}(G)$, is defined to be the minimum number of vertices necessary to delete in G in order to leave no component of order k or greater, i.e.,

$$\kappa_c^{(k)}(G) = \min \{ |X| \mid X \subset V(G) \text{ and } c(G - X) < k \} .$$

A survey of results can be found in [17]. Following [17], for a given k we will call a set X such that $c(G - X) < k$ a *failure set* of G for that k. We now have the following.

Theorem 2.5. *Let G be a graph with $u, v \in V(G)$ such that $uv \in E(G)$. Then $\kappa_c^{(k)}(G_{u\to v}) \leq \kappa_c^{(k)}(G)$ for any k.*

Proof. Let k be fixed and let X be a minimum order failure set of G for that k, so $|X| = \kappa_c^{(k)}$ and $c(G - X) < k$. By Theorem 2.2 there exists a set $X' \subset V(G)$ such that $|X'| = |X|$ and $c(G_{u\to v} - X') \leq c(G - X) < k$, and so deleting X' also leaves no component of $G_{u\to v} - X$ of order k or greater. Since $|X'| = |X|$, any minimum such failure set for k in $G_{u\to v}$ must be no larger than $|X|$, and therefore $\kappa_c^{(k)}(G_{u\to v}) \leq |X| = \kappa_c^{(k)}(G)$ as required. □

3 Compression and vertices at distance 2

In contrast to the previous results, for vertices $u, v \in V(G)$ that are not endpoints of an edge it is possible for compression from u to v to increase integrity, tenacity, or k-component order connectivity for some k. We now present an example of an infinite family of graphs where just such

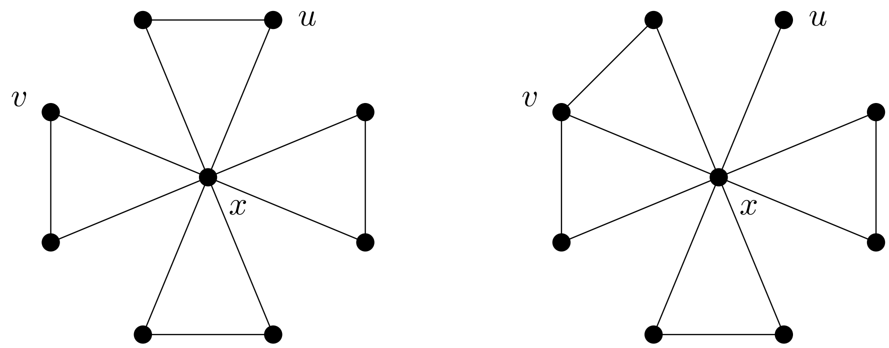

Figure 2: A graph H whose integrity, tenacity, and k-component order connectivity (for $k = 3$) increase after compression from u to v, when $d_H(u, v) = 2$. H is on the left, and $H_{u\to v}$ is on the right.

an increase occurs for vertices at distance 2. Since these parameters were just shown to decrease when u, v are adjacent, but increase here, to the best of our knowledge these are the first examples of graphs and graph parameters where the parameter does not uniformly increase or decrease after compression.

Example. The *join* of graphs G_1 and G_2, denoted $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{xy \,|\, x \in V(G_1) \text{ and } y \in V(G_2)\}$. Let $H = K_1 + sK_r$ for any $r > 1$ and $s > r + 1$, and let u and v be vertices from any two distinct K_r's. The graph $H_{u\to v}$ then consists of $K_1 + ((s-2)K_r \cup K_1 \cup C)$, where $C = K_1 + 2K_{r-1}$ and v is the cutvertex of C. An example of H and $H_{u\to v}$ for $r = 2$ and $s = 4$ appears in Figure 2.

Letting x denote the cutvertex of H and $H_{u\to v}$, we see the only I-set of H is the one-element set $\{x\}$, whose removal leaves s identical components each of order r, giving $I(G) = r + 1$. In $H_{u\to v}$, however, the I-set is the two-element set $\{x, v\}$ whose removal leaves $s - 2$ components of order r, two components of order $r - 1$, and the singleton u, which gives $I(H_{u\to v}) = r + 2 > I(H)$. For tenacity, it is not difficult to check that the T-sets are the same as the I-sets in H, which gives $T(H) = (r + 1)/s$ and $T(H_{u\to v}) = (r + 2)/(s + 1)$. Since $s > r + 1$, we have $T(H_{u\to v}) > T(H)$. And for $k = r + 1$, the same analysis shows that $\kappa_c^{(r+1)}(H_{u\to v}) = 2 > 1 = \kappa_c^{(r+1)}(H)$. ■

The example given, and in particular the I-sets, T-sets, and failure sets for $k = r+1$ that occur in the example, suggests examining the effects of adding v to the I-set, T-set, or failure set of G, which we do in the next theorem.

Theorem 3.1. *Let G be a connected graph with $u, v \in V(G)$, and let $X \subseteq V(G)$. If there does not exist an $X' \subseteq V(G_{u\to v})$ satisfying the three conditions of Theorem 2.2, then there exists an X'' such that all three of the following hold:*

1. $|X''| = |X| + 1$,
2. $\omega(G_{u\to v} - X'') = \omega(G - X) + 1$,
3. $c(G_{u\to v} - X'') \le c(G - X)$.

Proof. We will first establish under what conditions we do not have such an X' as in Theorem 2.2. If $uv \in E(G)$ then Theorem 2.2 holds, so $d_G(u,v) \geq 2$. Furthermore, we see that in Cases 1 and 3 of the proof of that theorem we did not require $uv \in E(G)$, and so those cases hold here as well. Thus if the X' of Theorem 2.2 does not exist we must have $u, v \notin X$. We consider the following three cases, the first two of which also show the three conditions of Theorem 2.2 are still satisfied, and the last of which gives the conditions shown above.

Case 1. $d_{G-X}(u,v) = 2$. Since u and v have a common neighbor in $G - X$ then they must be in the same component of $G - X$, and since compression does not change edges for common neighbors of u and v we must have u, v in the same component of $G_{u\to v} - X$ as well. Hence $G - X$ is identical to $G_{u\to v} - X$ with the exception that the component C containing u, v has been replaced in $G_{u\to v}$ by the component $C_{u\to v}$, and since $V(C_{u\to v}) = V(C)$ we may take $X' = X$ and the stronger inequalities of Theorem 2.2 hold.

Case 2. u, v in same component and $d_{G-X}(u,v) \geq 3$. As in the previous case $G - X$ is identical to $G_{u\to v} - X$ with the exception that C has been replaced with $C_{u\to v}$. Since $d_{G-X}(u,v) \geq 3$ here, however, $C_{u\to v}$ consists of two components, the isolated vertex u and another component whose vertex set is $V(C) - \{u\}$. By Lemma 2.1 the other components of $G_{u\to v} - X$ are identical to the non-u, v components of $G - X$, and so we must have $\omega(G_{u\to v} - X) > \omega(G - X)$ and $c(G_{u\to v} - X) \leq c(G - X)$, which means we can take $X' = X$ and the inequalities of Theorem 2.2 hold here again.

Case 3. u, v in different components. Call the components C_u and C_v respectively, and let $C_1, \ldots, C_k$ denote the components of $G - X$ other than C_u and C_v, so that

$$G - X = C_u \cup C_v \cup \bigcup_{i=1}^{k} C_i$$

with the unions above all disjoint unions. If either C_u or C_v are trivial, i.e., $C_u = u$ or $C_v = v$, then $G_{u\to v} - X = G - X$ and we may take $X' = X$ as in earlier cases. So we assume that C_u and C_v are non-trivial. Now as in the previous case in $G_{u\to v} - X$ we have u an isolated vertex, but now the component containing v, all it C_v', has a very specific structure: C_v' consists of two blocks with v as the cutvertex, and those blocks are isomorphic to C_u and C_v. Let $X'' = X \cup v$. Then

$$G_{u\to v} - X'' = G_{u\to v} - X - v = u \cup (C_u - u) \cup (C_v - v) \cup \bigcup_{i=1}^{k} C_i$$

with the unions above all disjoint unions. Since C_u, C_v are non-trivial, $\omega(G - X) + 1 = \omega(G_{u\to v} - X'')$ as required. In addition, clearly we have $c(G_{u\to v} - X'') \leq c(G - X)$, and since $|X''| = |X \cup v| = |X| + 1$, we have $|X''| + c(G_{u\to v} - X'') \leq |X| + c(G - X) + 1$ as required, completing the case and the proof. □

The previous theorem allows us to say more about compression's general effects on integrity, tenacity, and k-component order connectivity, and also permits us to determine compression's effects on rupture degree. For integrity and k-component order connectivity we have the following.

Theorem 3.2. *Let G be a graph with $u, v \in V(G)$. Then*

$$I(G_{u\to v}) \leq I(G) + 1$$

and

$$\kappa_c^{(k)}(G_{u\to v}) \leq \kappa_c^{(k)}(G) + 1$$

for any k. Furthermore if equality holds for the first (resp. second) inequality, then v is a member of some I-set (resp. failure set for k) of $G_{u\to v}$.

Proof. Let X be an I-set of G, so that $I(G) = |X| - c(G - X)$. By Theorems 2.2 and 3.1 there exists either a set X' such that $|X'| = |X|$ and $c(G_{u\to v} - X') \leq c(G - X)$, or a set X'' such that $|X''| = |X| + 1$ and $c(G_{u\to v} - X'') \leq c(G - X)$. If the former then

$$I(G_{u\to v}) \leq |X'| + c(G_{u\to v} - X') \leq |X| + c(G - X) = I(G)$$

and if the latter then

$$I(G_{u\to v}) \leq |X''| + c(G_{u\to v} - X') \leq |X| + c(G - X) + 1 = I(G) + 1$$

and so for all choices of u, v we have $I(G_{u\to v}) \leq I(G) + 1$. And similarly if X is a failure set for k of G, then either $k > c(G - X) \geq c(G_{u\to v} - X')$ or $k > c(G - X) \geq c(G_{u\to v} - X'')$. In the former case $\kappa_c^{(k)}(G_{u\to v}) \leq |X'| = |X| = \kappa_c^{(k)}(G)$ and in the latter case $\kappa_c^{(k)}(G_{u\to v}) \leq |X''| \leq |X| + 1 = \kappa_c^{(k)}(G) + 1$, and so for all choices of u, v we have $\kappa_c^{(k)}(G_{u\to v}) \leq \kappa_c^{(k)}(G) + 1$. Finally, as seen in the proof of Theorem 3.1, whenever the X'' appears it necessarily contains v. $\square$

For tenacity, we see that as long as tenacity is sufficiently high then compression behaves uniformly with respect to choice of vertices.

Theorem 3.3. *Let G be a graph such that $T(G) \geq 1$, and let $u, v \in V(G)$. Then $T(G_{u\to v}) \leq T(G)$.*

Proof. Let X be a T-set of G, so that $T(G) = (|X| + c(G - X))/\omega(G - X) \geq 1$, and note that this implies that

$$\frac{|X| + c(G - X) + 1}{\omega(G - X) + 1} \leq \frac{|X| + c(G - X)}{\omega(G - X)}.$$

By Theorems 2.2 and 3.1 there exists either a set X' such that $|X'| = |X|$ and $c(G_{u\to v} - X') \leq c(G - X)$ and $\omega(G_{u\to v} - X') \geq \omega(G - X)$, or a set X'' such that $|X''| = |X| + 1$ and $c(G_{u\to v} - X'') \leq c(G - X)$ and $\omega(G_{u\to v} - X'') = \omega(G - X) + 1$. If the former then

$$T(G_{u\to v}) \leq \frac{|X'| + c(G_{u\to v} - X')}{\omega(G_{u\to v} - X')} \leq \frac{|X| + c(G - X)}{\omega(G - X)} = T(G)$$

and if the latter then, since $T(G) \geq 1$,

$$T(G_{u\to v}) \leq \frac{|X''| + c(G_{u\to v} - X'')}{\omega(G_{u\to v} - X'')} \leq \frac{|X| + c(G - X) + 1}{\omega(G - X) + 1} \leq \frac{|X| + c(G - X)}{\omega(G - X)} = T(G)$$

as required. $\square$

The rupture degree of a graph G, denoted $r(G)$, was introduced in [22], and is defined to be

$$r(G) = \max\left\{\omega(G-X) - |X| - c(G-X) \mid X \subset V(G) \text{ and } \omega(G-X) > 1\right\}.$$

We call $X \subset V(G)$ an r-set if $r(G) = \omega(G-X) - |X| - c(G-X)$. Unusually, and in contrast to the vulnerability parameters mentioned so far (including the number of spanning trees and all-terminal reliability), $r(G)$ may take on negative values. In fact, as indicated by the fact that $r(K_n) = 1-n$, it is the smaller and negative values of $r(G)$ that indicate "stronger" graphs. Since the calculation of rupture degree involves the same quantities as the previous parameters we have the following result, which shows that compression weakens a graph in terms of rupture degree, and in fact does so for any choice of $u, v \in V(G)$.

Theorem 3.4. *Let G be a graph with $u, v \in V(G)$. Then $r(G_{u\to v}) \geq r(G)$.*

Proof. Let X be an r-set of G, so that $r(G) = \omega(G-X) - |X| - c(G-X)$. By Theorems 2.2 and 3.1 there exists either a set X' such that $|X'| = |X|$ and $c(G_{u\to v} - X') \leq c(G-X)$ and $\omega(G_{u\to v} - X') \geq \omega(G-X)$, or a set X'' such that $|X''| = |X| + 1$ and $c(G_{u\to v} - X'') \leq c(G-X)$ and $\omega(G_{u\to v} - X'') = \omega(G-X) + 1$. If the former then

$$r(G_{u\to v}) \geq \omega(G_{u\to v} - X') - |X'| - c(G_{u\to v} - X') \geq \omega(G-X) - |X| - c(G-X) = r(G)$$

and if the latter then

$$\begin{aligned}
r(G_{u\to v}) &\geq \omega(G_{u\to v} - X'') - |X''| - c(G_{u\to v} - X'') \\
&\geq (\omega(G-X) + 1) - (|X| + 1) - c(G-X) \\
&= \omega(G-X) - |X| - c(G-X) \\
&= r(G)
\end{aligned}$$

as required. □

4 Threshold and Quasi-Threshold Graphs

Threshold graphs are a well-known and much studied class of graphs and there are many equivalent ways to define them, see for example [23]. For our purposes, one of the more informative definitions involves a dominance relation on vertices. We say *vertex v dominates vertex u in G* if $N_G[u] \subseteq N_G[v]$, where $N_G[u]$ is the closed neighborhood $N_G[u] = N_G(u) \cup u$. A *threshold graph* is a graph in which, given any pair of vertices $u, v \in V(G)$, either u dominates v or v dominates u.

If $G_{u\to v} \neq G$ then the compression operation takes two vertices $u, v \in V(G)$ which do not dominate each other and produces a new graph $G_{u\to v}$ in which v does dominate u. After compression, then, a graph is "more threshold" and continued application of compression (with different pairs of vertices) can only increase this. Eventually, continuing the compression operation will result in a threshold graph, a fact noted originally by Bogdanowicz [6] and by Satyanarayana, Schoppmann, and Suffel [25], as well as by a number of authors since [8, 13, 15, 16]. Since compression decreases spanning trees and all-terminal reliability, it can therefore be concluded [10, 25] that for any connected graph G there is a connected threshold graph H such that $\tau(H) \leq \tau(G)$ and $Rel_p(H) \leq Rel_p(G)$. In other words, there are threshold graphs that minimize those parameters. In the same way, Theorem 3.4 implies the following.

Theorem 4.1. *For any connected graph G, there is a connected threshold graph H with the same number of vertices and edges such that $r(H) \geq r(G)$. In other words, there are threshold graphs that maximize rupture degree.*

An interesting question is which particular threshold graphs maximize rupture degree. Another useful way of defining threshold graphs are as particular kinds of *split graphs*, graphs whose vertex sets can be partitioned into two sets, one of which induces a clique and the other of which induces an independent set. Another way to define a threshold graph is as a split graph in which the vertices of the independent set have nested neighborhoods [23]. In [5] Boesch et al. conjectured the form of the particular threshold graph $L_{n,m}$ that would, for given $n = |V(G)|$ and $m = |E(G)|$, achieve the minimum values of $\tau(G)$ and $Rel_p(G)$. Informally, these are obtained by making the clique as large as possible and then, when connecting the vertices of the independent set to the clique, making as many degree one vertices as possible. Formally, let k be the least integer such that $m \geq \binom{n-k}{2} + k$. Then $L_{n,m}$ is the threshold graph consisting of an $(n-k)$-clique, with $k-1$ pendant vertices and one vertex of degree $m - \binom{n-k}{2} - k - 1$ attached to it. Confirming Boesch's conjecture that $L_{n,m}$ minimizes $\tau(G)$ and $Rel_p(G)$ for all p appears to be difficult. After about 20 years the conjecture was proven correct for $\tau(G)$ by Bogdanowicz [8], with the proof requiring a long and technical optimization argument. The conjecture that the $L_{n,m}$ graphs minimize all-terminal reliability remains open.

In a companion paper to this one [19], we showed that compression operates uniformly for the parameters toughness, edge toughness, binding number, and scattering number (which, like rupture degree, takes smaller and negative values for "stronger" graphs), and thus that there are threshold graphs that minimize (or maximize, in the case of scattering numbers) those parameters. We also conjectured there that the $L_{n,m}$ graphs are extremal graphs for those parameters. We conjecture the same is true for rupture degree.

Conjecture 4.2. *For any connected G with n vertices and m edges, $r(L_{n,m}) \geq r(G)$. In other words, the graph $L_{n,m}$ maximizes rupture degree over all connected graphs with n vertices and m edges.*

As the example given in Section 3 shows, however, for the parameters integrity, tenacity, and k-component order connectivity, compression behaves somewhat differently. For these parameters, compressions for $d_G(u,v) = 1$ and $d_G(u,v) = 2$ may have different effects, possibly decreasing the parameters in the first case and possibly increasing them in the second case.

A related class of graphs that may also be defined in terms of a dominance relation on vertices is the class of quasi-threshold graphs. A *quasi-threshold graph* is a graph in which, given any pair of vertices $u, v \in V(G)$ such that $uv \in E(G)$, either u dominates v or v dominates u. The class of quasi-threshold graphs properly contains the class of threshold graphs. (For instance the "bow-tie graph" $K_1 + 2K_2$ is quasi-threshold but not threshold.) Much as repeated compression can eventually result in a threshold graph, repeated compression using different pairs of adjacent vertices can eventually result in a quasi-threshold graph [15, 16]. As a consequence of this fact and Theorems 2.3, 2.4, and 2.5, we have the following.

Theorem 4.3. *For any graph G, there exists a quasi-threshold graph H such that $p(H) \leq p(G)$, where p is any of the parameters integrity, tenacity, or k-component order connectivity for any k.*

It is also an interesting question to ask which quasi-threshold graphs minimize these parameters. Since threshold graphs are a subclass of quasi-threshold graphs, it is possible that threshold graphs

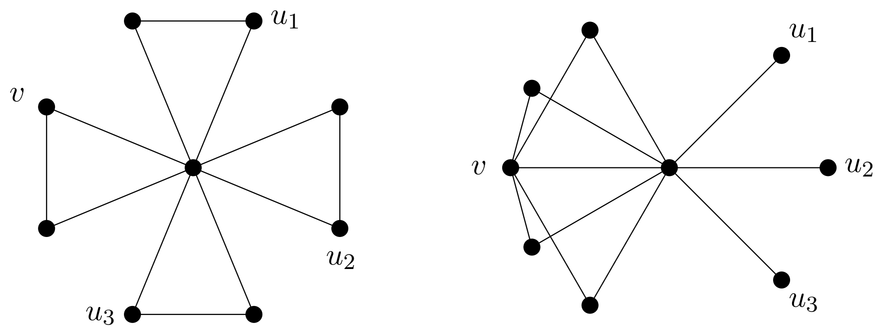

Figure 3: The graph H from the example of Section 3 on the left, and H after successive compressions from u_1, u_2, and u_3 to v on right. The graph on the right is threshold.

are still the minimizing graphs here as well. In this regard, we revisit our example from Section 3 earlier. Figure 3 shows the original graph H from the example, as well as the graph that results from successive compressions with u_1, u_2, and u_3 all compressed to v. The graph on the right, which we will call H', is threshold, and it is not difficult to check that $T(H') = 3/7$, which gives $T(H') < T(H) < T(H_{u\to v})$. Hence tenacity "reverses direction" during the three compressions, increasing after one compression but then decreasing to a minimum after additional compressions to a threshold graph. A similar "reversal" happens with integrity, with $3 = I(H') = I(H) < I(H_{u\to v}) = 4$. It is also worth noting that the graph H', while threshold, is not the threshold graph $L_{9,12}$ (which has K_4 as a subgraph) and that for integrity, tenacity, and k-component order connectivity (for any k) we have $p(H') \le p(L_{9,12})$, as is also easy to check. Hence if threshold graphs are indeed extremal for those three parameters, the extremal threshold graphs are not necessarily the graphs $L_{n,m}$.

Acknowledgement

The author would like to thank the reviewers for their many helpful comments which significantly improved the paper.

References

[1] Bagga, K.S., Beineke, L.W., Goddard, W.D., Lipman, M.J., Pippert, R.E., A survey of integrity. *Discret. Appl. Math.* 37 (1992), 13–28.

[2] C.A. Barefoot, R. Entringer, H. Swart, Vulnerability in graphs—a comparative survey. *J. Comb. Math. Comb. Comput.* 1 (1987), 12–22.

[3] F. Boesch, D. Gross, C. Suffel, Component order connectivity. Proceedings of the Twenty-ninth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1998). *Congr. Numer.* 131 (1998), 145–155.

[4] F. Boesch, D. Gross, C. Suffel, Component order connectivity—a graph invariant related

to operating component reliability. *Combinatorics, graph theory, and algorithms, Vol. I, II (Kalamazoo, MI, 1996)*, 109–116, New Issues Press, Kalamazoo, MI, 1999.

[5] F.T. Boesch, A. Satyanarayana, C.L. Suffel, Least reliable networks and reliability domination. *IEEE Trans. Commun.* 38 (1990) 2004–2009.

[6] Z. Bogdanowicz, Spanning trees in undirected simple graphs (Ph.D. Dissertation), Stevens Institute of Technology, New Jersey, USA, 1985.

[7] Z. Bogdanowicz, Chordal 2-connected graphs and spanning trees. *J. Graph Theory* 76 (2014), 224–235.

[8] Z. Bogdanowicz, Undirected simple connected graphs with minimum number of spanning trees. *Disc. Math.* 309 (2009), 3074–3082.

[9] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 2002.

[10] J. Brown, C. Colbourn and J. Devitt, Network transformations and bounding network reliability. *Networks* 23 (1993), 1–17.

[11] M.B. Cozzens, D. Moazzami, S. Stueckle, The tenacity of the Harary graphs, *J. Combin. Math. Combin. Comput.* 16 (1994), 33–56.

[12] M.B. Cozzens, D. Moazzami, S. Stueckle, The Tenacity of a Graph, in: Yousef Alavi, Allen Schwenk (Eds.), *Graph Theory, Combinatorics, and Algorithms*, Wiley, New York, 1995, pp. 1111-–1122.

[13] P. Csikvári, Applications of the Kelmans transformation: extremality of threshold graphs. *Electron. J. Combin.* 18 (2011), Paper 182, 24 pp.

[14] P. Csikvári, On a conjecture of V. Nikiforov. *Disc. Math.* 309 (2009), 4522–4526.

[15] J. Cutler, A.J. Radcliffe, Extremal graphs for homomorphisms. *J. Graph Theory* 67 (2011), 261–284.

[16] J. Cutler, A.J. Radcliffe, Extremal graphs for homomorphisms II. *J. Graph Theory* 76 (2014), 42–59.

[17] D. Gross, M. Heinig, L. Iswara, L.W. Kazmierczak, K. Luttrell, J.T. Saccoman, C. Suffel, A Survey of Component Order Connectivity Models of Graph Theoretic Networks. *WSEAS Transactions on Mathematics*, 12 (2013) 895–910.

[18] D. Gross, N. Kahl, and J.T. Saccoman, Graphs with the maximum or minimum number of k-factors. *Disc. Math.* 310 (2010), 687–691.

[19] N. Kahl, Graph vulnerability parameters, compression, and threshold graphs. Submitted.

[20] A.K. Kelmans, On graphs with randomly deleted edges. *Acta. Math. Acad. Sci. Hung.* 37 (1981), 77–88.

[21] L. Keough, A.J. Radcliffe, Graphs with the fewest matchings. *Combinatorica* 36 (2016), 703–723.

[22] Y. Li, S. Zhang, X. Li, Rupture degree of graphs. *Int. J. Comput. Math.* 82 (2005), 793-–803.

[23] N.V.R. Mahadev and U.N. Peled. Threshold Graphs and Related Topics. *Annals of Discrete Math. Vol. 56*, North-Holland Publishing Co., Amsterdam, 1995.

[24] D. Moazzami, Stability measure fo a graph: a survey. *Util. Math.* 57 (2000), 171–191.

[25] A. Satyanarayana, L. Schoppmann, C. Suffel, A reliability-improving graph transformation with applications to network reliability. *Networks* 22 (1992), 209–216.