Best Monotone Degree Conditions for Binding Number

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Abstract

We give sufficient conditions on the vertex degrees of a graph G to guarantee that G has binding number at least b, for any given $b > 0$. Our conditions are best possible in exactly the same way that Chvátal's well-known degree condition to guarantee a graph is hamiltonian is best possible.

1 Introduction

We consider only simple graphs without loops or multiple edges. Our terminology and notation will be standard except as indicated, and a good reference for any undefined terms or notation is [17]. We mention that for two graphs G, H on disjoint vertex sets, we will denote their *disjoint union* by $G \cup H$ and their *join* by $G + H$. Also, we will occasionally use G, rather than $V(G)$, to refer to the set of vertices of the graph G.

For a positive integer n, an *n-sequence* (or just a *sequence*) is an integer sequence $\pi = (d_1, d_2, \ldots, d_n)$, with $0 \leq d_j \leq n-1$ for all j. In contrast to [17], we will usually write the sequence in nondecreasing order, and may make this explicit by writing $\pi = (d_1 \leq \cdots \leq d_n)$. We will employ the standard abbreviated notation for sequences, e.g., $(4, 4, 4, 4, 4, 5, 5, 6)$ will be denoted $4^5\, 5^2\, 6^1$. If $\pi = (d_1, \ldots, d_n)$ and $\pi' = (d'_1, \ldots, d'_n)$ are two n-sequences, we say π' *majorizes* π, denoted $\pi' \geq \pi$, if $d'_j \geq d_j$ for all j.

A *degree sequence* of a graph is any sequence $\pi = (d_1, d_2, \ldots, d_n)$ consisting of the vertex degrees of the graph. A sequence π is *graphical* if there exists a graph G having π as one of its degree sequences, in which case we call G a *realization* of π. If P is a graph property (e.g., hamiltonian, k-connected, etc.), we call a graphical sequence π *forcibly P graphical* (or just *forcibly P*) if every realization of π has property P.

Historically, the degree sequence of a graph has been used to provide sufficient conditions for a graph to have certain properties, such as hamiltonian or k-connected. In particular, sufficient conditions for π to be forcibly hamiltonian were given by several authors, culminating in the following theorem of Chvátal [9].

Theorem 1.1. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If $d_i \leq i < \frac{n}{2}$ implies $d_{n-i} \geq n - i$, then π is forcibly hamiltonian.*

Unlike its predecessors, Chvátal's theorem has the property that if it does not guarantee that π is forcibly hamiltonian because the condition fails for some $i < \frac{n}{2}$, then π is majorized by $\pi' = i^i\,(n-i-1)^{n-2i}\,(n-1)^i$, which has a nonhamiltonian realization $K_i + (\overline{K_i} \cup K_{n-2i})$. As we will see below, this implies that Chvátal's theorem is the strongest of an entire class of theorems giving sufficient degree conditions for π to be forcibly hamiltonian.

A few years later, Boesch [5] recast, in the form of Theorem 1.2 below, an earlier sufficient condition of Bondy [6] for a degree sequence to be forcibly k-connected. He also showed the condition was strongest in exactly the same way as Chvátal's forcibly hamiltonian condition.

Theorem 1.2. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 2$, and let $1 \leq k \leq n-1$. If $d_i \leq i + k - 2$ implies $d_{n-k+1} \geq n - i$, for $1 \leq i \leq \frac{1}{2}(n-k+1)$, then π is forcibly k-connected.*

A method to obtain degree conditions for other graph properties, some as strong as Theorems 1.1 and 1.2, was described in [7].

A graph property P is called *increasing* if whenever a graph G has P, so does every edge-augmented supergraph of G. In particular, "hamiltonian" and "k-connected" are both increasing graph properties. In the remainder of this paper, the term "graph property" will always mean an increasing graph property.

Given a graph property P, consider a theorem T which declares certain degree sequences to be forcibly P, rendering no decision on the remaining degree sequences. We call such a theorem T a *forcibly P theorem* (or just a *P theorem*). Thus Theorem 1.1 is a forcibly hamiltonian theorem. We call a P theorem T *monotone* if, for any two degree sequences π, π', whenever T declares π forcibly P and $\pi' \geq \pi$, then T declares π' forcibly P. We call a P theorem T *optimal* (resp., *weakly-optimal*) if whenever T does not declare π forcibly P, then π has a realization without property P (resp., then there exists π', so that $\pi' \geq \pi$ and π' has a realization without property P). A P theorem which is both monotone and weakly-optimal is a best monotone P theorem in the following sense.

Theorem 1.3. *Let T, T_0 be monotone P theorems, with T_0 weakly-optimal. If T declares a degree sequence π to be forcibly P, then so does T_0.*

Proof of Theorem 1.3: Suppose to the contrary that there exists a degree sequence π that T declares forcibly P, but T_0 does not. Since T_0 is weakly-optimal, there exists a degree sequence $\pi' \geq \pi$ having a realization G' without property P; in particular, T will not declare π' forcibly P. But if T declares π forcibly P, $\pi' \geq \pi$, and T does not declare π' forcibly P, then T is not monotone, a contradiction. $\blacksquare$

Chvátal's hamiltonian theorem (Theorem 1.1) is clearly monotone, and we noted previously that it is weakly-optimal. So by Theorem 1.3, Chvátal's theorem is a best monotone hamiltonian theorem.

More recently, the problem of finding best monotone theorems has been considered for several other graph properties and parameters; e.g., toughness [1], existence of a 2-factor [2], independence number [3], chromatic number [3], and edge-connectivity [4]. In this note we continue this investigation by considering a best monotone theorem for the binding number of a graph.

Woodall introduced the binding number of a graph G in [18]. Given $S \subseteq V(G)$, let $N(S) \subseteq V(G)$ denote the neighbor set of S. Let $\mathcal{S} = \{S \subseteq V(G) \mid S \neq \emptyset \text{ and } N(S) \neq V(G)\}$. The *binding number* of G, denoted $\text{bind}(G)$, is defined by

$$\text{bind}(G) = \min_{S \in \mathcal{S}} \frac{|N(S)|}{|S|}.$$

A set $S \in \mathcal{S}$ for which the above minimum is attained will be called a *binding set* for G. For $b \geq 0$, we call a graph G *b-binding* if $\text{bind}(G) \geq b$. Cunningham [10] has shown that determining $\text{bind}(G)$ is tractable.

A number of theorems in the literature guarantee that a graph G has a given property if $\text{bind}(G)$ is bounded below by some value or function. Perhaps the best known such result is the following result of Woodall, where the constant $\frac{3}{2}$ is best possible [18, 19].

Theorem 1.4. *If G is a graph with* $\text{bind}(G) \geq \frac{3}{2}$, *then G is hamiltonian.*

Other graph properties that are guaranteed by lower bounds on $\text{bind}(G)$ include k-extendability [8, 15], containing a k-clique [11], and having certain types of factors [13, 14].

Our main goal in this paper is to establish a best monotone b-binding theorem for any $b > 0$. We do this in the next section, first when $0 < b \le 1$, and then when $b \ge 1$.

In the final section, we introduce a new perspective about sufficient degree conditions for graph properties. Suppose a graphical sequence π satisfies some (and thus, by Theorem 1.3, every) best monotone P theorem for a graph property P. We then call π *best monotone* P, denoted $\pi \in BM(P)$. We consider how best possible structural implications of the form P_1 implies P_2 can sometimes be improved, in degree terms, to $\pi \in BM(P)$ implies $\pi \in BM(P_2)$, where P is a substantially weaker property than P_1.

2 Best Monotone b-Binding Theorems, for $b > 0$

We begin with the best possible minimum degree condition for a graph to be b-binding, for any $b > 0$.

Theorem 2.1. *Let $b > 0$. If a graph G on $n \ge 1$ vertices satisfies $\delta(G) \ge \dfrac{bn}{b+1}$, then* $\text{bind}(G) \ge b$.

To see that Theorem 2.1 is best possible, consider $G = K_{\lceil \frac{bn}{b+1} \rceil - 1} + \overline{K_{\lfloor \frac{n}{b+1} \rfloor + 1}}$. Then $\delta(G) = \left\lceil \dfrac{bn}{b+1} \right\rceil - 1$, and taking $S = \overline{K_{\lfloor \frac{n}{b+1} \rfloor + 1}}$, we have

$$\text{bind}(G) \le \frac{|N(S)|}{|S|} = \frac{\lceil \frac{bn}{b+1} \rceil - 1}{\lfloor \frac{n}{b+1} \rfloor + 1} < \frac{\frac{bn}{b+1}}{\frac{n}{b+1}} = b.$$

Proof of Theorem 2.1: Let S be a binding set for G. If $\text{bind}(G) < b$, then $|S| > \dfrac{|N(S)|}{b} \ge \dfrac{\delta(G)}{b}$. Since $|S| > n - \delta(G)$ implies the contradiction $N(S) = V(G)$, we also have $|S| \le n - \delta(G)$. But then $\dfrac{\delta(G)}{b} < |S| \le n - \delta(G)$, or $\delta(G) < \dfrac{bn}{b+1}$, a contradiction. ■

If $\delta(G)$ fails to satisfy the condition in Theorem 2.1, we may still be able to conclude that G is b-binding by considering the full degree sequence of G. We show this by presenting the best monotone b-binding theorems below, first when $0 < b \le 1$ (Theorem 2.2), and then when $b \ge 1$ (Theorem 2.3). Each of these theorems is essentially a collection of conditions designed to block the degree sequences of certain key edge-maximal not-b-binding graphs. These graphs will be described explicitly in the paragraphs following the statements of the theorems. The sufficiency of blocking the degree sequences of just these key graphs is, of course, accomplished in the subsequent proofs. A fuller description of this approach for constructing best monotone theorems can be found in [1].

We first give a best monotone b-binding theorem for $0 < b \leq 1$.

Theorem 2.2. *Let $0 < b \leq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b+1 \rceil = 2$. If*

(i) $d_i \leq \lceil bi \rceil - 1 \implies d_{n-\lceil bi \rceil+1} \geq n-i$, *for* $1 \leq i \leq \left\lfloor \dfrac{n}{b+1} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{n}{b+1} \rfloor+1} \geq n - \left\lfloor \dfrac{n}{b+1} \right\rfloor$,

then π is forcibly b-binding.

Before proving Theorem 2.2, we show that it is best monotone b-binding. It is clearly monotone, and so by Theorem 1.3, it suffices to show that it is weakly optimal.

If π fails to satisfy condition (i) for some i, consider $\pi' = (\lceil bi \rceil - 1)^i (n-i-1)^{n-i-\lceil bi \rceil+1} (n-1)^{\lceil bi \rceil -1} \geq \pi$, with realization $G' = K_{\lceil bi \rceil -1} + \big(K_{n-i-\lceil bi \rceil+1} \cup \overline{K_i}\big)$. Taking $S = \overline{K_i}$, we find

$$\mathrm{bind}(G') \leq \frac{|N(S)|}{|S|} = \frac{\lceil bi \rceil - 1}{i} < \frac{bi}{i} = b.$$

Note also that condition (i) for index i explicitly blocks the degree sequence π'.

If π fails to satisfy condition (ii), consider $\pi' = \big(n - \lfloor \frac{n}{b+1} \rfloor - 1\big)^{\lfloor \frac{n}{b+1} \rfloor +1} (n-1)^{n-\lfloor \frac{n}{b+1} \rfloor -1} \geq \pi$, with realization $G' = K_{n-\lfloor \frac{n}{b+1} \rfloor -1} + \overline{K_{\lfloor \frac{n}{b+1} \rfloor +1}}$. Taking $S = \overline{K_{\lfloor \frac{n}{b+1} \rfloor +1}}$ we find

$$\mathrm{bind}(G') \leq \frac{|N(S)|}{|S|} = \frac{n - \lfloor \frac{n}{b+1} \rfloor - 1}{\lfloor \frac{n}{b+1} \rfloor + 1} < \frac{n - \frac{n}{b+1}}{\frac{n}{b+1}} = b.$$

Note that condition (ii) explicitly blocks the degree sequence π'.

Proof of Theorem 2.2: Suppose π satisfies (i) and (ii), but has a realization G with $\mathrm{bind}(G) < b$. Let $S \subseteq V(G)$ be a binding set for G, so that $\mathrm{bind}(G) = \dfrac{|N(S)|}{|S|} < b$. Define $A \doteq S - N(S)$, $B \doteq N(S) - S$, $C \doteq S \cap N(S)$, and $D \doteq V(G) - (S \cup N(S))$, so that $S = A \cup C$ and $N(S) = B \cup C$. Clearly, A is an independent set.

Since $\mathrm{bind}(G) = \dfrac{|B| + |C|}{|A| + |C|} < b \leq 1$, we have $|A| > |B|$, and so $A \neq \emptyset$. Also, $N(A) \subseteq B$, and so $N(A) \neq V(G)$. If $|C| > 0$, then

$$\mathrm{bind}(G) \leq \frac{|N(A)|}{|A|} \leq \frac{|B|}{|A|} < \frac{|B| + |C|}{|A| + |C|} = \mathrm{bind}(G),$$

a contradiction. Hence, $C = \emptyset$ and $\mathrm{bind}(G) = \dfrac{|B|}{|A|}$.

We consider two cases.

Case 1. $|A| \geq \left\lfloor \dfrac{n}{b+1} \right\rfloor + 1.$

Then $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \leq d_{|A|} \leq |B| = n - |A| - |D| \leq n - \lfloor \frac{n}{b+1} \rfloor - 1$, contradicting condition (ii).

Case 2. $|A| \leq \left\lfloor \dfrac{n}{b+1} \right\rfloor.$

Since $\dfrac{|B|}{|A|} = \text{bind}(G) < b$, we have $n - |D| = |A| + |B| < |A| + b|A| \leq n$, or $D \neq \emptyset$. So each vertex in A has degree at most $n - |A| - |D| \leq n - |A| - 1$, while each vertex in D has degree at most $|B| + |D| - 1 = n - |A| - 1$. Thus $d_{|A|+|D|} \leq n - |A| - 1$. Set $i \doteq |A|$, so $1 \leq i \leq \left\lfloor \dfrac{n}{b+1} \right\rfloor$. Since $|B| < b|A|$, we have $|B| \leq \lceil b|A| \rceil - 1$. But then

$$d_i = d_{|A|} \leq |B| \leq \lceil b|A| \rceil - 1 = \lceil bi \rceil - 1,$$

while

$$d_{n - \lceil bi \rceil - 1} = d_{n - \lceil b|A| \rceil + 1} \leq d_{n-|B|} = d_{|A|+|D|} \leq n - |A| - 1 = n - i - 1,$$

contradicting condition (i). ■

Next, we give a best monotone b-binding theorem for $b \geq 1$, which is identical to Theorem 2.2 when $b = 1$.

Theorem 2.3. *Let $b \geq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b + 1 \rceil$. If*

(i) $d_i \leq n - \left\lfloor \dfrac{n-i}{b} \right\rfloor - 1 \implies d_{\lfloor \frac{n-i}{b} \rfloor + 1} \geq n - i$, *for* $1 \leq i \leq \left\lfloor \dfrac{n}{b+1} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \dfrac{n}{b+1} \right\rfloor$,

then π is forcibly b-binding.

Before proving Theorem 2.3, we show that it is best monotone b-binding. Clearly it is monotone, and so, by Theorem 1.3, it suffices to show that it is weakly optimal. If π fails to satisfy condition (i) for some i, consider $\pi' = \left(n - \lfloor \frac{n-i}{b} \rfloor - 1\right)^i (n - i - 1)^{\lfloor \frac{n-i}{b} \rfloor - i + 1} (n-1)^{n - \lfloor \frac{n-i}{b} \rfloor - 1} \geq \pi$, with realization $G' = K_{n - \lfloor \frac{n-i}{b} \rfloor - 1} + \left(K_{\lfloor \frac{n-i}{b} \rfloor - i + 1} \cup \overline{K_i}\right)$. Taking $S = \left(K_{\lfloor \frac{n-i}{b} \rfloor - i + 1} \cup \overline{K_i}\right)$, we find

$$\text{bind}(G') \leq \frac{|N(S)|}{|S|} = \frac{n-i}{\lfloor \frac{n-i}{b} \rfloor + 1} < \frac{n-i}{\left(\frac{n-i}{b}\right)} = b.$$

Note that condition (i) for index i explicitly blocks the degree sequence π'.

If π fails to satisfy condition (ii), we may argue exactly as when condition (ii) failed in Theorem 2.2.

To prove Theorem 2.3, we need the following.

Lemma 2.4. *If π satisfies conditions (i) and (ii) in Theorem 2.3 for some $b \geq 1$, then π is forcibly 1-binding.*

Proof of Lemma 2.4: To show π is forcibly 1-binding, it suffices, by Theorem 2.2 with $b = 1$, to show

(1) $\quad d_i \leq i - 1 \implies d_{n-i+1} \geq n - i$, for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, and

(2) $\quad d_{\lfloor \frac{n}{2} \rfloor + 1} \geq n - \left\lfloor \frac{n}{2} \right\rfloor$,

For (1), if $1 \leq i \leq \left\lfloor \frac{n}{b+1} \right\rfloor$, then notice that by condition (i) in Theorem 2.3, $d_i \leq i - 1 \leq n - \left\lfloor \frac{n-i}{b} \right\rfloor - 1$ implies $d_{n+i-1} \geq d_{\lfloor \frac{n-i}{b} \rfloor + 1} \geq n - i$, which is (1). However, if $\left\lfloor \frac{n}{b+1} \right\rfloor + 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, then condition (ii) in Theorem 2.3 gives

$$d_i \geq d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor \geq n - (i-1) > i,$$

and (1) is vacuously satisfied.

For (2), note that by condition (ii) in Theorem 2.3 we have

$$d_{\lfloor \frac{n}{2} \rfloor + 1} \geq d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor \geq n - \left\lfloor \frac{n}{2} \right\rfloor,$$

which is (2). Thus, π is forcibly 1-binding. ■

Proof of Theorem 2.3: Suppose π satisfies (i) and (ii), but has a realization G with $\text{bind}(G) < b$. Let $S \subseteq V(G)$ be a largest binding set in G, so that $\text{bind}(G) = \frac{|N(S)|}{|S|} < b$. Partition $V(G)$ into $A \doteq S - N(S)$, $B \doteq N(S) - S$, $C \doteq S \cap N(S)$, and $D \doteq V(G) - (S \cup N(S))$, so that $S = A \cup C$ and $N(S) = B \cup C$. Clearly, A is an independent set.

Claim. $|C| \geq |D|$

Proof of Claim: Suppose $|D| > |C|$. Define $S' \doteq A \cup D$, so $N(S') \subseteq B \cup D$. Since $N(S) = B \cup C \neq V(G)$, we have $S' \neq \emptyset$. Since $S = A \cup C \neq \emptyset$, we also have $N(S') \neq V(G)$. Therefore, since $|D| > |C|$ and π is forcibly 1-binding,

$$1 \leq \text{bind}(G) \leq \frac{|N(S')|}{|S'|} \leq \frac{|B| + |D|}{|A| + |D|} \leq \frac{|B| + |C|}{|A| + |C|} = \frac{|N(S)|}{|S|} = \text{bind}(G),$$

and S' is a binding set in G. However, $|D| > |C|$ implies $|S'| > |S|$, contradicting our choice of S. This proves the Claim. $\square$

Note that each vertex in A has degree at most $|B| = n - (|A| + |C| + |D|)$, and each vertex in D has degree at most $|B| + |D| - 1 = n - (|A| + |C| + 1)$. Therefore, since $|A| + |D| \geq 1$,

$$d_{|A|+|D|} \leq n - (|A| + |C|) \leq n - (|A| + |D|). \tag{1}$$

Also, each vertex in C has degree at most $|B| + |C| - 1 = n - (|A| + |D| + 1)$, so

$$d_{|A|+|C|} \leq n - (|A| + |D| + 1), \text{ if } |C| \geq 1. \tag{2}$$

Case 1. $|A| + |D| \geq \left\lfloor \dfrac{n}{b+1} \right\rfloor + 1.$

By (1), $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \leq d_{|A|+|D|} \leq n - (|A| + |D|) < n - \left\lfloor \dfrac{n}{b+1} \right\rfloor$, contradicting condition (ii).

Case 2. $|A| + |D| \leq \left\lfloor \dfrac{n}{b+1} \right\rfloor.$

Note that $|C| \geq 1$ (else $|D| = |C| = 0$ by the Claim, and $b > \dfrac{|N(S)|}{|S|} = \dfrac{n - |A|}{|A|}$, or $|A| > \dfrac{n}{b+1}$, contradicting the Case).

Since $\dfrac{|N(S)|}{|S|} < b$, we have

$$|A| + |C| = |S| > \frac{|N(S)|}{b} = \frac{n - (|A| + |D|)}{b},$$

or

$$|A| + |C| \geq \left\lfloor \frac{n - (|A| + |D|)}{b} \right\rfloor + 1. \tag{3}$$

Set $i \doteq |A| + |D|$, so $1 \leq i \leq \lfloor \frac{n}{b+1} \rfloor$. By (1) and (3),

$$d_i = d_{|A|+|D|} \leq n - (|A| + |C|) \leq n - \left\lfloor \frac{n - (|A| + |D|)}{b} \right\rfloor - 1 = n - \left\lfloor \frac{n-i}{b} \right\rfloor - 1,$$

while by (2) and (3),

$$d_{\lfloor \frac{n-i}{b} \rfloor + 1} = d_{\lfloor \frac{n-(|A|+|D|)}{b} \rfloor + 1} \leq d_{|A|+|C|} < n - (|A| + |D|) = n - i.$$

This contradicts condition (i). $\blacksquare$

3 Best Monotone Degree Improvement of Theorem 1.4

If a graphical sequence π satisfies a best monotone P theorem for a graph property P, we call π *best monotone* P, and denote this by $\pi \in BM(P)$. For example, $\pi = 4^6 \in BM(\text{hamiltonian})$, since π satisfies Theorem 1.1 (Chvátal's theorem). Our goal in this section is to see how implications of the form $\pi \in BM(P_1)$ implies $\pi \in BM(P_2)$ reflect, and occasionally improve, implications of the form P_1 implies P_2.

Let P_1, P_2 be two graph properties. If P_1 implies P_2 and $\pi \in BM(P_1)$, then π is forcibly P_2. However, we can say more.

Theorem 3.1. *If P_1, P_2 are graph properties such that P_1 implies P_2, then for any graphical sequence π we have $\pi \in BM(P_1)$ implies $\pi \in BM(P_2)$.*

Proof of Theorem 3.1: Suppose to the contrary that $\pi \in BM(P_1)$, but $\pi \notin BM(P_2)$. Then there exists a graphical sequence $\pi' \geq \pi$ having a realization G' without property P_2. Since P_1 implies P_2, G' cannot have property P_1. However, $\pi \in BM(P_1)$ and $\pi' \geq \pi$ together imply that $\pi' \in BM(P_1)$, and thus every realization of π' has P_1, a contradiction. ■

Taking P_1 to be '$\frac{3}{2}$-binding' and P_2 to be 'hamiltonian', we know P_1 implies P_2 by Theorem 1.4. So by Theorem 3.1

$$\pi \in BM\left(\frac{3}{2} - \text{binding}\right) \text{ implies } \pi \in BM(\text{hamiltonian}). \tag{4}$$

We may think of (4) as a best monotone degree analogue of Theorem 1.4.

As we have noted, the constant $\frac{3}{2}$ in Theorem 1.4 is best possible. However, the constant $\frac{3}{2}$ in (4) can be substantially improved.

Theorem 3.2. *Let $b > 1$. Then for any graphical sequence π, $\pi \in BM$(b-binding) implies $\pi \in BM$(hamiltonian).*

Note that every hamiltonian graph is necessarily 1-binding, and thus by Theorem 3.1, $\pi \in BM(\text{hamiltonian})$ implies $\pi \in BM(\text{1-binding})$. On the other hand, the converse does not hold. To see this consider

$$\pi = \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)^{\lfloor \frac{n}{2} \rfloor - 1} \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right)^{n - 2\lfloor \frac{n}{2} \rfloor + 2} (n-1)^{\lfloor \frac{n}{2} \rfloor - 1}$$

with realization $K_{\lfloor \frac{n}{2} \rfloor -1} + \left(\overline{K_{\lfloor \frac{n}{2} \rfloor -1}} \cup K_{n-2\lfloor \frac{n}{2} \rfloor +2} \right)$. It is easily verified that $\pi \in BM$(1-binding), while $\pi \notin BM$(hamiltonian) since π fails to satisfy Theorem 1.1 for $i = \left\lfloor \frac{n}{2} \right\rfloor - 1$. Thus $b > 1$ in Theorem 3.2 is best possible.

Proof of Theorem 3.2: Suppose $\pi \in BM(b\text{-binding})$ for some $b > 1$ and $d_i \leq i$ for some $i < \frac{n}{2}$. We will show that $d_{n-i} \geq n - i$, so that π satisfies Theorem 1.1, and thus $\pi \in BM$(hamiltonian). We consider two cases.

Case 1. $i \leq \left\lfloor \frac{n}{b+1} \right\rfloor$.

If $i \geq n - \left\lfloor \frac{n-i}{b} \right\rfloor \geq n - \frac{n-i}{b}$, then $i \geq n$ since $b > 1$, a contradiction. Hence $i \leq n - \left\lfloor \frac{n-i}{b} \right\rfloor - 1$. Then $d_i \leq i \leq n - \left\lfloor \frac{n-i}{b} \right\rfloor - 1$ for $i \leq \left\lfloor \frac{n}{b+1} \right\rfloor$, and thus by Theorem 2.3(i),

$$d_{n-i} \geq d_{n-\left(n-\lfloor \frac{n-i}{b} \rfloor -1\right)} = d_{\lfloor \frac{n-i}{b} \rfloor +1} \geq n - i,$$

as required.

Case 2. $\left\lfloor \frac{n}{b+1} \right\rfloor + 1 \leq i < \frac{n}{2}$.

Then

$$d_{\lfloor \frac{n}{b+1} \rfloor +1} \leq d_i \leq i < \frac{n}{2} = n - \frac{n}{2} < n - \left\lfloor \frac{n}{b+1} \right\rfloor,$$

which contradicts Theorem 2.3(ii). Therefore, no such i exists with $d_i \leq i$. ■

We call a graph G on $n \geq 3$ vertices *pancyclic* if G contains an l-cycle for each l such that $3 \leq l \leq n$. In [16], Shi generalized Theorem 1.4 as follows.

Theorem 3.3. *If G is a graph with bind$(G) \geq \frac{3}{2}$, then G is pancyclic.*

Since the constant $\frac{3}{2}$ is best possible in Theorem 1.4, it is a fortiori best possible in Theorem 3.3.

We have the following best monotone condition for a degree sequence to be forcibly pancyclic.

Theorem 3.4. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If*

(1) $d_i \leq i < \frac{n}{2} \Longrightarrow d_{n-i} \geq n - i$, *and*

(2) $d_n \geq \frac{n}{2} + 1$, *if n is even,*

then π is forcibly pancyclic.

Before proving Theorem 3.4, we show that it is best monotone pancyclic. It is clearly monotone, and so by Theorem 1.3, it suffices to show it is weakly optimal. If (1) fails for some $i < \frac{n}{2}$, then π is majorized by the degrees of the nonhamiltonian (nonpancyclic) graph $K_i + (\overline{K_i} \cup K_{n-2i})$. If (2) fails, then π is majorized by the degrees of the bipartite (nonpancyclic) graph $K_{\frac{n}{2},\frac{n}{2}}$.

Proof of Theorem 3.4: In [12] it was shown that if π satisfies (1), then every realization of π is either pancyclic or bipartite. However, if a realization of π were bipartite, and necessarily hamiltonian by Theorem 1.1, then n is even and $d_n \leq \frac{n}{2}$, contradicting (2). ■

We now prove a theorem which relates to Theorem 3.3 precisely as Theorem 3.2 relates to Theorem 1.4.

Theorem 3.5. *Let $b > 1$. Then for any graphical sequence π, $\pi \in BM$(b-binding) implies $\pi \in BM$(pancyclic).*

Proof of Theorem 3.5: If $\pi \in BM$(b-binding) with $b > 1$, then $\pi \in BM$(hamiltonian) by Theorem 3.2. Thus π satisfies Theorem 1.1, which is (1) in Theorem 3.4. So it suffices to show π also satisfies (2) in Theorem 3.4.

Since $b > 1$ and $\pi \in BM$(b-binding), π satisfies condition (ii) in Theorem 2.3. Thus we obtain

$$d_n \geq d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor \geq n - \frac{n}{b+1} > n - \frac{n}{2} = \frac{n}{2}.$$

So, $d_n \geq \frac{n}{2} + 1$ when n is even, which is (2) in Theorem 3.4. ■

It would be interesting to explore other graph properties P_1, P_2 such that P_1 implies P_2 is best possible, but the corresponding relation $\pi \in BM(P_1)$ implies $\pi \in BM(P_2)$, guaranteed by Theorem 3.1, can be improved as above.

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