Improving Theorems in a Best Monotone Sense

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Abstract

We demonstrate how certain theorems can be improved when we know the degree sequence of a graph. In particular, let $\tau(G)$ and $\text{bind}(G)$ be the toughness and binding number, respectively, of a graph G. We show how a recent best possible lower bound on $\tau(G)$ in terms of $\text{bind}(G)$, when $\text{bind}(G) \geq 2$, can be improved when combined with a knowledge of the degree sequence of G.

1 Introduction

A *degree sequence* of a graph G is a list of the degrees of all the vertices of G, with repetition if multiple vertices have the same degree. In this paper the degree sequences are in nondecreasing order. If π is a degree sequence of length n, then we typically denote it as $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$. At times we may utilize exponents to indicate the number of times a degree appears, e.g., $\pi = (2, 2, 2, 2, 4) = 2^4 4^1$. Given two sequences $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ and $\pi' = (d'_1 \leq d'_2 \leq \cdots \leq d'_n)$, we say that π' *majorizes* π, denoted $\pi' \geq \pi$, if $d'_i \geq d_i$ for all i. A sequence $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ is a *graphical sequence* if there exists a graph G with π as its degree sequence, and we then call G a *realization* of π. A graphical sequence π can have more than one distinct realization. If every realization of π has property P, we say that π is *forcibly* P. For example, the graphical sequence $\pi = 3^6$ is forcibly hamiltonian.

Historically, the degree sequence of a graph has been used to provide sufficient conditions for a graph to have certain properties, such as hamiltonicity or k-connectedness. In particular, sufficient conditions for π to be forcibly hamiltonian were given by several authors, culminating in the following theorem of Chvátal [10].

Theorem 1.1. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq 3$. If $d_i \leq i < \frac{n}{2} \implies d_{n-i} \geq n-i$, then π is forcibly hamiltonian.*

Unlike its predecessors, Chvátal's theorem has the property that if it does not guarantee that π is forcibly hamiltonian because the condition fails for some $i < \frac{n}{2}$, then π is majorized by $\pi' = i^i\,(n-i-1)^{n-2i}\,(n-1)^i$, which has a nonhamiltonian realization $K_i + (\overline{K_i} \cup K_{n-2i})$. As we will see below, this implies that Chvátal's theorem is the strongest of an entire class of theorems giving sufficient degree conditions for π to be forcibly hamiltonian.

A few years later, Boesch [7] recast, in the form of Theorem 1.2 below, an earlier sufficient condition of Bondy [8] for a degree sequence to be forcibly k-connected. He also showed the condition was strongest in exactly the same sense as Chvátal's forcibly hamiltonian condition.

Theorem 1.2. *Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence with $n \geq 2$, and let $1 \leq k \leq n-1$. If $d_i \leq i+k-2 \implies d_{n-k+1} \geq n-i$, for $1 \leq i \leq \frac{1}{2}(n-k+1)$, then π is forcibly k-connected.*

A method to obtain degree conditions for other graph properties, some as strong as Theorems 1.1 and 1.2, was described in [9].

A graph property P is called *ancestral* if whenever a graph G has P, so does every edge-augmented supergraph of G. In particular, "hamiltonian" and "k-connected" are both ancestral graph properties. In the remainder of this paper, the term "graph property" will always mean an ancestral graph property.

Given a graph property P, consider a theorem T which declares certain degree sequences to be forcibly P, rendering no decision on the remaining degree sequences. We call such a theorem T a *forcibly P theorem* (or just a *P theorem*). Thus Theorem 1.1 is a forcibly hamiltonian theorem. We call a P theorem T *monotone* if, for any two degree sequences π, π', whenever T declares π forcibly P and $\pi' \geq \pi$, then T declares π' forcibly P. We call a P theorem T *optimal* (resp., *weakly optimal*) if whenever T does not declare π forcibly P, then π has a realization without property P (resp., then there exists π', so that $\pi' \geq \pi$ and π' has a realization without property P). In view of the following result, a P theorem which is both monotone and weakly optimal is called a *best monotone P theorem*.

Theorem 1.3. *Let T, T_0 be monotone P theorems, with T_0 weakly optimal. If T declares a degree sequence π to be forcibly P, then so does T_0.*

Proof. Suppose to the contrary that there exists a degree sequence π that T declares forcibly P, but T_0 does not. Since T_0 is weakly optimal, there exists a degree sequence $\pi' \geq \pi$ having a realization G' without property P; in particular, T will not declare π' forcibly P. But if T declares π forcibly P, $\pi' \geq \pi$, and T does not declare π' forcibly P, then T is not monotone, a contradiction. ■

Theorem 1.1 is clearly monotone, and in [9] it is shown that it is weakly optimal. Thus Theorem 1.1 is best monotone with respect to the property of hamiltonicity.

More recently, the problem of finding best monotone theorems has been considered for several other graph properties and parameters; e.g., toughness [1], existence of a 2-factor [2], edge-connectivity [3], independence number [4], chromatic number [4], and binding number [6].

The main result of this paper concerns the graph parameters toughness and binding number. Chvátal introduced the notion of the toughness of a graph in [10]. Let $\omega(G)$ denote the number of components of a graph G. For $t \geq 0$, we call G *t-tough* if $t \cdot \omega(G - X) \leq |X|$ for every $X \subseteq V(G)$ with $\omega(G - X) \geq 2$. The *toughness* of G, denoted $\tau(G)$, is the maximum $t \geq 0$ for which G is t-tough, so that

$$\tau(G) = \min \left\{ \frac{|X|}{\omega(G - X)} \,\middle|\, X \subset V(G) \text{ and } \omega(G - X) \geq 2 \right\}.$$

By convention, $\tau(K_n) := n - 1$. If G is not complete, we call $X \subseteq V(G)$ a *tough set* of G if $\omega(G - X) \geq 2$ and $\tau(G) = |X|/\omega(G - X)$.

In [11], Woodall introduced the notion of the binding number of a graph G. If $S \subseteq V(G)$, let $N(S)$ denote the set of neighbors of S in G, including any vertices of S that have neighbors in S. For $b \geq 0$, we call G *b-binding* if $b|S| \leq |N(S)|$ for all $S \subseteq V(G)$ with $N(S) \neq V(G)$. The *binding number* of G, denoted $\text{bind}(G)$, is the maximum $b \geq 0$ such that G is b-binding. Thus,

$$\text{bind}(G) = \min \left\{ \frac{|N(S)|}{|S|} \,\middle|\, \emptyset \neq S \subseteq V(G),\ N(S) \neq V(G) \right\}.$$

In particular, $\text{bind}(K_n) = n - 1$. We call $S \subseteq V(G)$ a *binding set* of G if $N(S) \neq V(G)$ and $\text{bind}(G) = |N(S)|/|S|$.

2 Improvements in a Best Monotone Sense

If a graphical sequence π satisfies some (and thus, by Theorem 1.3, every) best monotone P theorem for a graph property P, we call π *best monotone* P, and denote this by $\pi \in \mathrm{BM}(P)$. For example, $\pi = 4^6 \in \mathrm{BM}(\text{hamiltonian})$, since π satisfies Theorem 1.1 (Chvátal's theorem). We now explore how implications of the form $\pi \in \mathrm{BM}(P_1) \implies \pi \in \mathrm{BM}(P_2)$ reflect, and occasionally improve, implications of the form $P_1 \implies P_2$.

Let P_1, P_2 be two graph properties. If $P_1 \implies P_2$ and $\pi \in \mathrm{BM}(P_1)$, then π is forcibly P_2. However, we can say more.

Theorem 2.1. *If P_1, P_2 are graph properties such that $P_1 \implies P_2$, then for every graphical sequence π, $\pi \in \mathrm{BM}(P_1) \implies \pi \in \mathrm{BM}(P_2)$.*

Proof. Suppose to the contrary that $\pi \in \mathrm{BM}(P_1)$ but $\pi \notin \mathrm{BM}(P_2)$. Then there exists a graphical sequence $\pi' \geq \pi$ having a realization G' without property P_2. Since $P_1 \implies P_2$, G' cannot have property P_1. However, $\pi \in \mathrm{BM}(P_1)$ and $\pi' \geq \pi$, which together imply that $\pi' \in \mathrm{BM}(P_1)$, and thus every realization of π' has P_1, a contradiction. ∎

As an example of the above, consider the following theorem of Woodall [11].

Theorem 2.2. *If G is a graph with $\mathrm{bind}(G) \geq \frac{3}{2}$, then G is hamiltonian.*

A best monotone theorem for the property of b-binding, for $b \geq 1$, is given below as it appears in [6].

Theorem 2.3. *Let $b \geq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \lceil b+1 \rceil$. If*

(i) $d_j \leq n - \left\lfloor \frac{n-j}{b} \right\rfloor - 1 \implies d_{\lfloor \frac{n-j}{b} \rfloor + 1} \geq n - j$, *for* $1 \leq j \leq \left\lfloor \frac{n}{b+1} \right\rfloor$, *and*

(ii) $d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor$,

then π is forcibly b-binding.

Taking P_1 to be '$\frac{3}{2}$-binding' and P_2 to be 'hamiltonian', we know $P_1 \implies P_2$ by Theorem 2.2. So by Theorem 2.1

$$\pi \in \mathrm{BM}\left(\tfrac{3}{2}\text{-binding}\right) \implies \pi \in \mathrm{BM}(\text{hamiltonian}). \qquad (1)$$

In other words, if π satisfies the conditions of Theorem 2.3 when $b = 3/2$, then π is guaranteed to satisfy the conditions of Theorem 1.1. We may think of (1) as a best monotone degree analogue of Theorem 2.2.

It is well known that the constant 3/2 in Theorem 2.2 is best possible. However, in [6] it is shown that the constant 3/2 in (1) can be substantially improved.

Theorem 2.4. *Let $b > 1$. Then for any graphical sequence π, $\pi \in \mathrm{BM}(b\text{-}binding) \implies \pi \in \mathrm{BM}(hamiltonian)$.*

It is also shown in [6] that the condition $b > 1$ in Theorem 2.4 is best possible. To see this, consider

$$\pi = \left(\left\lfloor \frac{n}{2} \right\rfloor - 1\right)^{\lfloor \frac{n}{2} \rfloor - 1} \left(n - \left\lfloor \frac{n}{2} \right\rfloor\right)^{n-2\lfloor \frac{n}{2} \rfloor + 2} (n-1)^{\lfloor \frac{n}{2} \rfloor - 1}$$

with realization $K_{\lfloor \frac{n}{2} \rfloor - 1} + \left(\overline{K_{\lfloor \frac{n}{2} \rfloor - 1}} \cup K_{n-2\lfloor \frac{n}{2} \rfloor + 2}\right)$. It is easily verified that $\pi \in \mathrm{BM}(\text{1-binding})$, while $\pi \notin \mathrm{BM}(\text{hamiltonian})$ since π fails to satisfy Theorem 1.1 for $i = \lfloor \frac{n}{2} \rfloor - 1$.

Consider a theorem T_1 of the form $P_1 \implies P_2$, where P_1 and P_2 are two graph properties. We call a theorem T_2 an *improvement of T_1 in a best monotone sense* if T_2 has the form $\pi \in \mathrm{BM}(P_1') \implies \pi \in \mathrm{BM}(P_2')$, where $P_1 \implies P_1'$, $P_2' \implies P_2$, and either $P_1' \not\Longrightarrow P_1$ or $P_2 \not\Longrightarrow P_2'$. Thus Theorem 2.4 is an improvement of Theorem 2.2 in a best monotone sense. We are especially interested in finding improvements of T_1 in a best monotone sense when T_1 is known to be best possible, e.g., as in Theorem 2.2.

Our goal for the remainder of this section is to provide an improvement in a best monotone sense of Theorem 2.5 below.

Theorem 2.5. [5, Theorem 1.7] *Let G be a graph with $\mathrm{bind}(G) \geq 2$. Then*

$$\tau(G) \geq \begin{cases} 3/2 & \textit{if } \mathrm{bind}(G) = 2, \\ 2 & \textit{if } \mathrm{bind}(G) = 9/4 \textit{ or } 2 + 1/(2m-1) \textit{ for some } m \geq 2, \\ 2 + 1/m & \textit{if } \mathrm{bind}(G) = 2 + 2/(2m-1) \textit{ for some } m \geq 2, \\ \mathrm{bind}(G) & \textit{otherwise.} \end{cases}$$

Moreover, these bounds are sharp for every possible value of $\mathrm{bind}(G) \geq 2$.

Our improvement of Theorem 2.5 in a best monotone sense requires a best monotone theorem for toughness. This is given below as it appears in [1].

Theorem 2.6. *Let $t \geq 1$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, with $n \geq \lceil t \rceil + 2$. If*

$$d_{\lfloor i/t \rfloor} \leq i \implies d_{n-i} \geq n - \lfloor i/t \rfloor, \textit{ for } t \leq i < \frac{tn}{t+1},$$

then π is forcibly t-tough.

We now derive the following improvement of Theorem 2.5 in a best monotone sense.

Theorem 2.7. *Let $b \geq 2$, and let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence. Then $\pi \in \mathrm{BM}(b\text{-binding}) \implies \pi \in \mathrm{BM}(b\text{-tough})$.*

Proof. Assume $\pi \in \mathrm{BM}(b\text{-binding})$ for some $b \geq 2$. It is proved in [6] that Theorem 2.3 is a best monotone theorem for the property of b-binding; that is, it characterizes the set BM(b-binding). Thus π must satisfy (i) and (ii) of Theorem 2.3.

Assume there exists i, with $b \leq i < \frac{bn}{b+1}$, such that $d_{\lfloor \frac{i}{b} \rfloor} \leq i$. Define $j := \lfloor \frac{i}{b} \rfloor$, so that $1 \leq j \leq \left\lfloor \frac{n}{b+1} \right\rfloor$. Our goal is to show that $d_{n-i} \geq n-j$, and thus π satisfies the hypotheses of Theorem 2.6 with $t = b$, i.e., $\pi \in \mathrm{BM}(b\text{-tough})$.

Case 1. $j = \left\lfloor \dfrac{n}{b+1} \right\rfloor$.

Since $j \leq \frac{i}{b} < \frac{n}{b+1}$, it follows that $\frac{n}{b+1} \notin \mathbb{Z}$ and

$$i \leq \left\lfloor \frac{bn}{b+1} \right\rfloor = \left\lfloor n - \frac{n}{b+1} \right\rfloor = n - \left\lfloor \frac{n}{b+1} \right\rfloor - 1. \qquad (2)$$

By (2) and Theorem 2.3(ii), it follows that

$$d_{n-i} \geq d_{n-\lfloor \frac{bn}{b+1} \rfloor} = d_{\lfloor \frac{n}{b+1} \rfloor + 1} \geq n - \left\lfloor \frac{n}{b+1} \right\rfloor = n - j.$$

Case 2. $1 \leq j \leq \left\lfloor \dfrac{n}{b+1} \right\rfloor - 1$.

In this case we will prove that

$$i \leq n - \left\lfloor \frac{n-j}{b} \right\rfloor - 1. \qquad (3)$$

Together with Theorem 2.3(i) and the fact that $d_j = d_{\lfloor \frac{i}{b} \rfloor} \leq i$, this will imply that $d_{n-i} \geq d_{\lfloor \frac{n-j}{b} \rfloor + 1} \geq n - j$, as required. Thus it suffices to prove (3).

Case 2a. $\dfrac{bn}{b+1} - 1 < i < \dfrac{bn}{b+1}$.

Then $i = \left\lfloor \frac{bn}{b+1} \right\rfloor$ and $\frac{bn}{b+1} \notin \mathbb{Z}$. Since $\frac{bn}{b+1} = n - \frac{n}{b+1}$, it follows that $\frac{n}{b+1} \notin \mathbb{Z}$ and (2) holds with equality. Also, $\frac{i}{b} > \frac{n}{b+1} - 1$, and so, by the hypothesis of Case 2, $j = \left\lfloor \frac{n}{b+1} \right\rfloor - 1$. Since (2) holds with equality, $i = n - j - 2 \geq n - j - b$, so that

$$\left\lfloor \frac{n-j}{b} \right\rfloor \leq \left\lfloor \frac{i+b}{b} \right\rfloor = j + 1 = n - i - 1.$$

This proves (3) in this case.

Case 2b. $i \leq \dfrac{bn}{b+1} - 1.$

Then $(b^2 - 1)(i+1) < b(b-1)n$, and $b < b^2 - 1$ since $b \geq 2$, and so

$$bi - j < bi - \frac{i}{b} + 1 = \frac{(b^2-1)i + b}{b} < \frac{(b^2-1)(i+1)}{b} < (b-1)n.$$

Thus $i < \frac{(b-1)n+j}{b} = n - \frac{n-j}{b}$. Since i and n are both integers, (3) holds in this case too. ∎

We conclude by showing that the lower bound of 2 for b in Theorem 2.7 is best possible. To see this, consider the degree sequence

$$\pi = (2m-3)^{m-2}(2m-2)^2(3m-4)^{2m-3},$$

for $m \geq 2$, which has realization $G = K_{2m-3} + \left(K_2 \cup \overline{K_{m-2}}\right)$.

Claim. $\pi \in \mathrm{BM}$ *(b-binding) and* $\pi \notin \mathrm{BM}$ *(b-tough), for* $b = \dfrac{2m-1}{m}$.

Proof. Let $b = \frac{2m-1}{m}$ and note that $n = 3m - 3$. Thus

$$\left\lfloor \frac{n}{b+1} \right\rfloor = \left\lfloor \frac{m(3m-3)}{3m-1} \right\rfloor = \left\lfloor \frac{3m(m-1)}{3m-1} \right\rfloor = m - 1 + \left\lfloor \frac{m-1}{3m-1} \right\rfloor = m - 1,$$

and

$$\frac{bn}{b+1} = \frac{(2m-1)(3m-3)}{3m-1} = 2m - 3 + \frac{2m}{3m-1}.$$

We first show that π satisfies the hypotheses of Theorem 2.3. Note that the range on j in Theorem 2.3(i) is $1 \leq j \leq m-1$. If $j \leq m-2$, then $n - j \geq 2m - 1$ and

$$n - \left\lfloor \frac{n-j}{b} \right\rfloor - 1 \leq 3m - 4 - \left\lfloor \frac{m(2m-1)}{2m-1} \right\rfloor = 2m - 4.$$

Therefore, $d_j > n - \lfloor \frac{n-j}{b} \rfloor - 1$ since the minimum degree in π is $2m-3$. Similarly, when $j = m-1$ we have

$$n - \left\lfloor \frac{n-j}{b} \right\rfloor - 1 = 2m - 3 < 2m - 2 = d_{m-1} = d_j.$$

Therefore, π satisfies Theorem 2.3(i) vacuously.

To see that Theorem 2.3(ii) is satisfied, note that

$$d_{\lfloor \frac{n}{b+1} \rfloor + 1} = d_m = 2m - 2 = (3m-3) - (m-1) = n - \left\lfloor \frac{n}{b+1} \right\rfloor.$$

Therefore, $\pi \in$ BM(b-binding).

On the other hand, $\tau(G) = \frac{2m-3}{m-1} < \frac{2m-1}{m} = b$. Thus π is not forcibly $\frac{2m-1}{m}$–tough, and $\pi \notin$ BM(b-tough). ∎

Since such an example exists for every $m \geq 2$, we see that 2 is the best possible lower bound on b for which Theorem 2.7 holds.

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