Graph Vulnerability Parameters, Compression, and Threshold Graphs

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Abstract

Given a simple graph G and vertices $u, v \in V(G)$, let $N_G(u)$ and $N_G(v)$ denote the neighborhoods in G of u and v respectively. The compression of G from u to v produces a new graph $G_{u \to v}$ by, for each $x \in N_G(u) - N_G(v) - \{v\}$, removing edges from G of the form ux and replacing them with corresponding edges of the form vx. Kelmans, and independently Satyanarayana, Schoppmann, and Suffel, showed that for any graph G and any $u, v \in V(G)$, compression from u to v could not increase, and typically decreased, both the number of spanning trees and the all-terminal reliability of G. Both the number of spanning trees and all-terminal reliability are vulnerability parameters, i.e., measures of the strength of a network. We show that a number of other prominent vulnerability parameters—including vertex connectivity, toughness, scattering number, edge connectivity, edge toughness, and binding number—are affected by compression in the same way as number of spanning trees and all-terminal reliability. As a consequence, as with the number of spanning trees and the all-terminal reliability, threshold graphs are extremal graphs for all of the vulnerability parameters considered.

1 Introduction

Let G be a graph and let $u, v \in V(G)$, and let $N_G(u)$ and $N_G(v)$ denote the neighborhoods in G of u and v respectively. The *compression of G from u to v* produces a new graph $G_{u \to v}$ by, for each $x \in N_G(u) - N_G(v) - \{v\}$, removing all edges from G of the form ux and replacing them with corresponding edges of the form vx. An illustration of this operation appears in Figure 1.

The compression operation—which has also been called a 'shift transformation' [4, 7], a 'Kelmans transformation' [11, 12], or, in its inverse form, a 'swing surgery' [23]—was first employed by Kelmans [19], and later independently by Satyanarayana, Schoppmann, and Suffel [23], both of whom showed that $\tau(G_{u \to v}) \le \tau(G)$ and $Rel_{G_{u \to v}}(p) \le Rel_G(p)$, where $\tau(G)$ and $Rel_G(p)$ represent respectively the number of spanning trees and the all-terminal reliability polynomial $Rel_G(p)$ of a graph G. Subsequently Brown, Colbourn, and Devitt [7] showed that $c_i(G_{u \to v}) \le c_i(G)$, where c_i denotes the i^{th} coefficient of the all-terminal reliability polynomial $Rel_G(p)$, generalizing both results. (The number of spanning trees is one of the coefficients of the polynomial.)

Compression has since been shown to affect a number of other graph parameters as well. In what follows we will indicate by "decreases parameter p" the relationship $p(G_{u \to v}) \le p(G)$, and in a similar manner use "increases." Compression has been shown to decrease the number of k-factors for every k [15], decrease the number of k-matchings for any k [21], increase the spectral

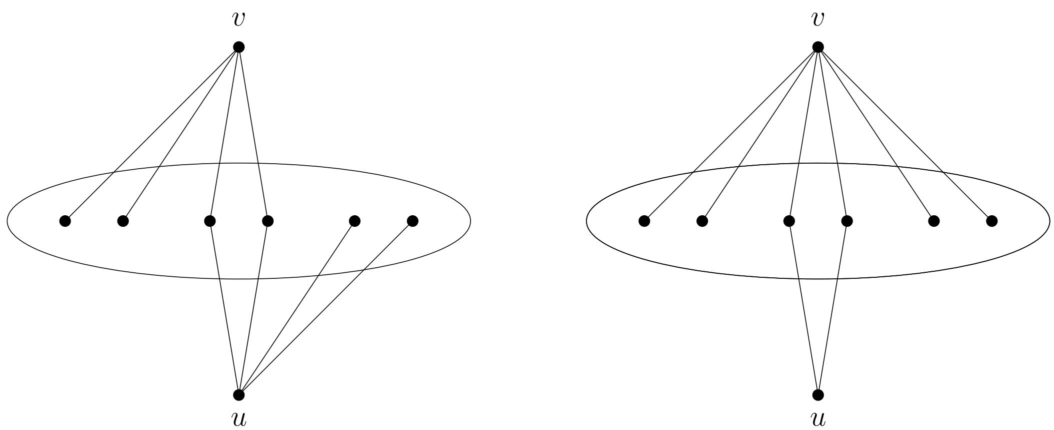

Figure 1: An illustration of compression. On the left is a graph G, and on the right $G_{u\to v}$, the compression of G from u to v. If edge uv had been present in G, it would also be present in $G_{u\to v}$.

radius of both G and its complement [12], increase the largest root of the matching polynomial [11], decrease the smallest real root of the independence polynomial and the coefficients of the chromatic polynomial [11], increase the number of independent sets of order k for any k [11, 13], and decrease a number of parameters associated with the Laplacian polynomial of a graph [20]. Most recently in [13, 14], whose "compression" terminology and $G_{u\to v}$ notation we follow, the result on number of independent sets was significantly generalized when compression from u to v was shown to increase the number of homomorphisms into certain target image graphs.

Both the number of spanning trees and the all-terminal reliability are examples of vulnerability parameters, measures that help gauge how strong a graph or network is. In this context Kelmans and Satyanarayana, Schoppmann, and Suffel have shown that compression can only weaken a graph. In this paper we look at the vulnerability parameters vertex connectivity, toughness, scattering number, edge connectivity, edge toughness, and binding number. For all of these parameters except scattering number, lower values of the parameter indicate weaker graphs, much like $\tau(G)$ and $Rel_G(p)$; for scattering number higher values indicate weaker graphs. In this paper we show that, by these vulnerability measures as well, compression can only weaken a graph. We accomplish this by determining compression's effects on the more basic measures on which the parameters are based, namely $\omega(G-X)$ and $\omega(G-F)$, the number of components of G when a vertex set X or an edge set F is removed, and $|N_G(S)|$, the size of the neighborhood of a vertex set S.

These results on compression in turn imply that extremal graphs for these parameters fall into the well-known class of *threshold graphs*. Threshold graphs may be defined in a number of different ways [22]; later in this paper two in particular will prove useful in addressing the extremal question. For now we simply note that Satyanarayana, Schoppmann, and Suffel and a number of authors since have demonstrated that given any graph G, it is possible to produce a threshold graph H from G via a sequence of compression operations, and if G is connected we may also take H to be a connected graph as well. As a consequence, in [23] it was shown that threshold graphs minimize the number of spanning trees and the all-terminal reliability of connected graphs. By the same reasoning, in this paper we obtain that threshold graphs minimize, or in the case of scattering number maximize, the vulnerability parameters considered here for connected graphs.

All graphs in this paper are assumed to be simple (although see [7] for an extension of compression to multigraphs). The vulnerability parameters considered, as well as the terms and notation necessary for those definitions, will be defined in the paper as they appear. For any undefined terms we refer the reader to a standard reference like [5]. In general our notation will be standard as well, although we note that when no confusion will arise we may omit brackets around vertex sets, for example writing $N_G(u,v)$ instead of $N_G(\{u,v\})$ or $X - v$ instead of $X - \{v\}$.

Finally, if $N_G(u) - N_G(v) - v$ is empty then $G_{u\to v} = G$, and if $N_G(v) - N_G(u) - u$ is empty then $G_{u\to v}$ is isomorphic to G, with the isomorphism obtained by simply switching the labels of u and v. In these instances we clearly have $p(G_{u\to v}) \leq p(G)$ as needed, so in the rest of the paper we assume that $N_G(u) - N_G(v) - v$ and $N_G(v) - N_G(u) - u$ are both non-empty.

2 Connectivity, Toughness, and Scattering Number

A number of vulnerability measures, including most of the ones considered in this paper, are concerned with what happens when vertices and/or edges are deleted from a graph. We begin then with some simple but useful observations about this situation which for reference we give as lemmas. To do so we require some notation. While the vertex set of G remains unchanged under compression, the edge set of $G_{u\to v}$ differs from that of G. It is possible, then, that edges of the set F mentioned in the lemmas (and in theorems of later sections) may not exist in $G_{u\to v}$. There does, of course, exist a natural correspondence between the missing edges of G and the new edges appearing in $G_{u\to v}$. By $F_{u\to v}$ we indicate F with any edges of the form ux replaced with the corresponding edges of the form vx, for any $x \in N_G(u) - N_G(v) - v$. We begin with the following straightforward observation which we give without proof.

Lemma 2.1. *Let G be a graph and $u, v \in V(G)$, with $W \subset V(G)$ and $F \subset E(G)$. If $u, v \in W$, then $(G - W) - F = (G_{u\to v} - W) - F_{u\to v}$.*

Note that, since any subgraph may be considered the result of vertex deletions followed by edge deletions, the lemma above also implies that a subgraph of G on a vertex set that excludes u, v, is also a subgraph in $G_{u\to v}$ on the same vertex set.

Lemma 2.2. *Let G be a graph and $u, v \in V(G)$, with $W \subset V(G)$ and $F \subset E(G)$. If C is a component of $(G-W)-F$ such that $u, v \notin V(C)$, then C is also a component of $(G_{u\to v}-W)-F_{u\to v}$ on the same vertex set.*

Proof. By the previous lemma since $u, v \notin V(C)$ the subgraph C on the same vertex set appears unchanged in $(G_{u\to v} - W) - F_{u\to v}$ and we only need to show that there is no $x \in V(C)$ and $y \notin V(C)$ such that xy is an edge of $(G_{u\to v} - W) - F_{u\to v}$. Assume to the contrary that such an edge exists. Since $xy \notin (G - W) - F$, then we must have either that xy was deleted, i.e. $y \in W$ or $xy \in F$, or that xy was moved during compression, i.e. $v = y$. In the former case clearly xy is also deleted in $(G_{u\to v} - W) - F_{u\to v}$ as well, a contradiction. And in the latter case if vx is an edge of $(G_{u\to v} - W) - F_{u\to v}$ this implies ux was an edge of $(G - W) - F$. Then in $(G - W) - F$ we had $u \in V(C)$, a contradiction as well. □

Let $\omega(G)$ denote the number of components of a graph G. We now give in effect the main theorem of the section, which essentially shows that $\omega(G - X)$ is in a sense monotonic with respect to compression.

Theorem 2.3. *Let G be a connected graph and $u, v \in V(G)$. Then for any fixed k,*

$$\max_{\substack{X \subseteq V(G),\\ |X|=k}} \omega(G - X) \leq \max_{\substack{X \subseteq V(G_{u\to v}),\\ |X|=k}} \omega(G_{u\to v} - X). \tag{1}$$

Proof. It suffices to show that for every $X \subseteq V(G)$ there exists an $X' \subseteq V(G_{u\to v})$ such that

1. $|X'| = |X|$, and
2. $\omega(G_{u\to v} - X') \geq \omega(G - X)$.

We consider the following four cases. In the first three cases we show that compression cannot decrease the number of components created by deleting the vertices of X, and so may take $X' = X$. In the final case a simple alteration of X produces the requisite X'.

Case 1. $v \in X$. In this case note that $G_{u\to v} - v$ is the subgraph of $G - v$ created by the additional deletion of edges of the form ux for $x \in N_G(u) - N_G(v)$. Thus $G_{u\to v} - X \subseteq G - X$, which implies that $\omega(G_{u\to v} - X) \geq \omega(G - X)$.

Case 2. $u, v \notin X$ and u, v in the same component of $G - X$. Call this component C. Then by Lemma 2.2 if C' is any component of $G - X$ other than C then C' appears unchanged in $G_{u\to v} - X$. Thus when comparing the quantities $\omega(G_{u\to v} - X)$ and $\omega(G - X)$ we may ignore all non-u, v components and assume $G - X = C$. But then clearly $1 = \omega(C) = \omega(G - X) \leq \omega(G_{u\to v} - X)$, completing the case.

Case 3. $u, v \notin X$ and u, v in different components of $G - X$. As in the previous case, by Lemma 2.2 we may ignore any non-u, v components, so here it suffices to consider the case where $G - X$ consists of two components C_u and C_v, which contain u and v respectively, and $\omega(G - X) = 2$. Since u, v are in separate components of $G - X$, we must have $uv \notin E(G)$ and $N_G(u) \cap N_G(v) \subseteq X$. But in $G_{u\to v}$ the only vertices possibly adjacent to u are v if $uv \in E(G)$ and the vertices of $N_G(u) \cap N_G(v)$. Hence in $G_{u\to v} - X$ we have u an isolate, and therefore $\omega(G_{u\to v} - X) \geq 2 = \omega(G - X)$, completing the case.

Case 4. $u \in X$ and $v \notin X$. Let $X' = X - u \cup v$. Then $G_{u\to v} - X'$ is isomorphic to a subgraph of $G - X$; to see this note that $G_{u\to v} - v$ is isomorphic to a subgraph of $G - u$ by simply changing the label u to the label v in $G_{u\to v} - v$. But $G_{u\to v} - X'$ isomorphic to a subgraph of $G - X$ implies $\omega(G - X) \leq \omega(G_{u\to v} - X')$. Finally, for this X' clearly $u \in X', v \notin X'$ and so $|X'| = |X - u \cup v| = |X|$. □

The *toughness* of G, first defined by Chvátal in [8] and denoted here by $t(G)$, is $t(K_n) = n - 1$ for the complete graph K_n, and otherwise

$$t(G) = \min \left\{ \frac{|X|}{\omega(G - X)} \,\middle|\, X \subset V(G) \text{ and } \omega(G - X) \geq 2 \right\}.$$

The *scattering number* of a graph G, denoted $sc(G)$, is related to toughness; in fact when introduced by Jung in [17], he referred to it as the "algebraic dual" of toughness. The scattering number of G is defined to be $sc(K_n) = 2 - n$ for the complete graph K_n, and otherwise

$$sc(G) = \max \left\{ \omega(G - X) - |X| \mid X \subset V(G) \text{ and } \omega(G - X) \geq 2 \right\}.$$

As an immediate algebraic consequence of the definitions and Theorem 2.3 we have the following.

Theorem 2.4. *Let G be a graph and $u, v \in V(G)$. Then $t(G_{u\to v}) \leq t(G)$.*

Theorem 2.5. *Let G be a connected graph on at least three vertices and $u, v \in V(G)$. Then $sc(G_{u\to v}) \geq sc(G)$.*

Finally, the compression result for connectivity $\kappa(G)$ follows almost as easily.

Theorem 2.6. *Let G be a graph with $u, v \in V(G)$. Then $\kappa(G_{u\to v}) \leq \kappa(G)$.*

Proof. Let X be a minimum cutset of G, i.e., the minimum order set $X \subset V(G)$ such that $\omega(G - X) \geq 2$. By Theorem 2.3 there exists an $X' \subseteq V(G_{u\to v})$ with $|X'| = |X|$ such that $2 \leq \omega(G-X) \leq \omega(G_{u\to v} - X')$, and thus this set X' disconnects $G_{u\to v}$ as well. Since $|X'| = |X|$, any minimum cutset of $G_{u\to v}$ must be no larger than $|X|$, and so $\kappa(G_{u\to v}) \leq |X| = \kappa(G)$ as required. □

3 Edge Connectivity and Edge Toughness

An analogue of Theorem 2.3 holds for edge sets. While our goal here is the removal of edge sets that split the graph G into multiple components, it will be useful to first prove the case of edge cutsets that split a connected G into exactly 2 components, i.e., minimal edge cuts.

Theorem 3.1. *Let G be a connected graph with $u, v \in V(G)$. Then for every minimal edge cut $F \subseteq E(G)$ there exists an edge cut $F' \subseteq E(G_{u\to v})$ such that*

1. *$|F'| \leq |F|$, and*
2. *$\omega(G_{u\to v} - F') \geq \omega(G - F) = 2$.*

Proof. We denote the distance between u and v in G by $d_G(u, v)$. If $d_G(u, v) \geq 3$, then it is easy to see that u is an isolate in $G_{u\to v}$. Then $\omega(G_{u\to v}) \geq 2 = \omega(G - F)$, and hence the theorem is satisfied with $F' = \emptyset$. We may therefore assume that $d_G(u, v) \leq 2$. We now consider the following cases. In the first case we show that we may take $F' = F_{u\to v}$, and in each of the remaining cases we show that isolating u in $G_{u\to v}$ more efficiently disconnects the graph.

Case 1. u, v in the same component C of $G - F$. Call the non-u, v component C'. Then by Lemma 2.2, C' must also appear identically in $G_{u\to v} - F_{u\to v}$. Thus $\omega(G_{u\to v} - F_{u\to v}) \geq 2 = \omega(G-F)$ and, since $|F_{u\to v}| = |F|$, taking $F' = F_{u\to v}$ works in this case.

In the remaining cases we have u, v in separate components, call them C_A and C_B respectively, and without loss of generality we take $v \in V(C_A)$ and $u \in V(C_B)$. Let A (resp. B) denote the vertices in C_A (resp. C_B) that are incident to the edge cut F, and note that since $d_G(u, v) \leq 2$, this means that either $v \in A$ or $u \in B$, or both. The following cases are therefore exhaustive.

Case 2. $v \in A, u \in B$. If $uv \in E_G(A, B)$ then $uv \in E_{G_{u\to v}}(A, B)$ as well, so when comparing the sizes of F and F' in this case we may assume that uv is not present. Now consider $x \in$

$N_G(u) \cap N_G(v)$. If $x \in A$, then we must have $xu \in F$ and $xv \notin F$, and if $x \in B$ then we must have $xv \in F$ and $xu \notin F$. Hence

$$\begin{aligned}|F| &\geq |A \cap (N_G(u) \cap N_G(v))| + |B \cap (N_G(u) \cap N_G(v))| \\ &= |(A \cup B) \cap (N_G(u) \cap N_G(v))| \\ &= |(A \cup B) \cap (N_{G_{u\to v}}(u) \cap N_{G_{u\to v}}(v))|\end{aligned}$$

which, since in this case $N_{G_{u\to v}}(u) \cap N_{G_{u\to v}}(v) \subseteq A \cup B$,

$$\begin{aligned}&= |N_{G_{u\to v}}(u) \cap N_{G_{u\to v}}(v)| \\ &= deg_{G_{u\to v}}(u).\end{aligned}$$

Therefore if we let F' be the set of edges incident to u in $G_{u\to v}$, or equivalently the edges between u and the vertices of $N_G(u) \cap N_G(v)$, by the above we have $|F'| \leq |F|$, and in $G_{u\to v} - F'$ we have u an isolate and thus $\omega(G_{u\to v} - F') \geq 2 = \omega(G - F)$.

Case 3. $v \in A, u \notin B$. In this case then $N_G(u) \cap N_G(v) \subseteq B$, and similarly as in the previous case we have

$$\begin{aligned}|F| &\geq |B \cap (N_G(u) \cap N_G(v))| \\ &= |B \cap (N_{G_{u\to v}}(u) \cap N_{G_{u\to v}}(v))| \\ &= |N_{G_{u\to v}}(u) \cap N_{G_{u\to v}}(v)| \\ &= deg_{G_{u\to v}}(u)\end{aligned}$$

and so here again setting F' equal to the edges of $G_{u\to v}$ incident to u will satisfy the conditions of the theorem.

Case 4. $v \notin A, u \in B$. In this case then $N_G(u) \cap N_G(v) \subseteq A$, and the calculations of the previous case are identical except with A replacing B. Again setting F' equal to the edges of $G_{u\to v}$ incident to u will satisfy the conditions of the theorem, completing the case and the proof. □

Since edge connectivity $\lambda(G)$ is defined in terms of minimal edge cuts, Theorem 3.1 already gives the following analogue to Theorem 2.6.

Theorem 3.2. *Let G be a graph with $u, v \in V(G)$. Then $\lambda(G_{u\to v}) \geq \lambda(G)$.*

Proof. Let F be a minimum edge cut of G, i.e. the minimum size set $F \subset E(G)$ such that $\omega(G - F) \geq 2$. For such a minimum edge cut we have $\omega(G - F) = 2$ and by Theorem 3.1 there exists an $F' \subset F_{u\to v}$ such that $|F'| \leq |F|$ and $\omega(G_{u\to v} - F_{u\to v}) \geq \omega(G - F) = 2$. Thus this edge set disconnects $G_{u\to v}$ as well. Since $|F'| \leq |F|$, any minimum edge cutset of $G_{u\to v}$ must be no larger than $|F|$, and so $\lambda(G_{u\to v}) \leq |F| = \lambda(G)$ as required. □

The next theorem effectively generalizes Theorem 3.1 to edge cuts creating more than two components.

Theorem 3.3. *Let G be a connected graph and $u, v \in V(G)$. Then for any fixed k,*

$$\max_{\substack{F \subseteq E(G),\\ |F| \leq k}} \omega(G - F) \leq \max_{\substack{F \subseteq E(G_{u\to v}),\\ |F| \leq k}} \omega(G_{u\to v} - F). \tag{2}$$

Proof. It suffices to show that for any edge cut $F \subseteq E(G)$ there exists an edge cut $F' \subseteq E(G_{u\to v})$ such that

1. $|F'| \leq |F|$, and
2. $\omega(G_{u\to v} - F') \geq \omega(G - F)$.

If u, v are in the same component of $G - F$ then the theorem follows as in Case 1 of Theorem 3.1, so we only consider when u, v are in separate components. Call these components C_u and C_v respectively. By Lemma 2.2, then, we have that any non-C_u, C_v components in $G - F$ remain unchanged in $G_{u\to v} - F_{u\to v}$, and we may ignore these components and assume that $G - F = C_u \cup C_v$. Now if G is connected, then in this case Theorem 3.1 applies and we are done. Thus we have that G is a disconnected graph and u and v are in separate components of G.

Call these components G_u and G_v. Now for G of this form, $G_{u\to v}$ has a very specific structure: $G_{u\to v}$ consists of two components, one the isolate u, and another component consisting of G_u and G_v but with u and v identified; in other words, the second component consists of two blocks with v the cutvertex, one block identical to G_v and the other block identical to G_u but with v replacing u. $G_{u\to v} - F_{u\to v}$ then consists of the isolate u and the components of $G_u - F$ and $G_v - F$, with the exception that the components C_u and C_v appear as one component consisting of C_u and C_v but with u and v identified. So

$$\omega(G_{u\to v} - F_{u\to v}) = 1 + (\omega(C_u - F) + \omega(C_v - F) - 1) = \omega(C_u - F) + \omega(C_v - F) = \omega(G - F)$$

and thus taking $F' = F_{u\to v}$ will work here, completing the proof. $\square$

The edge-toughness parameter was introduced by Gusfeld in [16] as an edge analogue of Chvátal's toughness parameter, and is defined as

$$\tau_e(G) = \min\left\{ \frac{|F|}{\omega(G - F) - 1} \,\middle|\, F \subset E(G) \text{ and } \omega(G - F) \geq 2 \right\}.$$

The edge toughness compression result is an immediate algebraic consequence of the definition and Theorem 3.3.

Theorem 3.4. *Let G be a graph with $u, v \in V(G)$. Then $\tau_e(G_{u\to v}) \geq \tau_e(G)$.*

4 Binding Number

In [24], Woodall introduced the notion of the binding number of a graph G. If $S \subseteq V(G)$, let $N(S)$ denote the set of neighbors of S in G, including any vertices of S that have neighbors in S. The *binding number* of G, denoted $\mathrm{bind}(G)$, is defined to be

$$\mathrm{bind}(G) = \min\left\{ \frac{|N(S)|}{|S|} \,\middle|\, \emptyset \neq S \subseteq V(G),\ N(S) \neq V(G) \right\}.$$

Somewhat surprisingly, to the best of our knowledge binding number is the only prominent vulnerability parameter which takes into account the quantity $|N_G(S)|$. As with the previous sections rather than simply proving a binding number result we first prove a more general theorem on the quantity $|N_G(S)|$ which then implies the binding number result. We begin with two simple but useful lemmas.

Lemma 4.1. *Let $S \subset V(G)$ be such that $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$. Then $v \in S$ or $S \cap (N_G(u) - N_G(v) - v) \neq \emptyset$, or both. Moreover, if $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$ and $S \cap (N_G(u) - N_G(v) - v) = \emptyset$ then*

$$N_{G_{u\to v}}(S) \subseteq N_G(S) \cup (N_G(u) - N_G(v) - v)$$

and if $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$ and $v \notin S$ then

$$N_{G_{u\to v}}(S) \subseteq N_G(S) \cup v.$$

Proof. If $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$, then there exists some vertex in $N_{G_{u\to v}}(S)$ that is not in $N_G(S)$. The only vertices that acquire new neighbors after compression from u to v are v itself and the vertices of $N_G(u) - N_G(v) - v$, so we must have either $v \in S$ or $S \cap (N_G(u) - N_G(v) - v) \neq \emptyset$. And now if $S \cap (N_G(u) - N_G(v) - v) = \emptyset$ then necessarily $v \in S$, and the only new neighbors acquired by v are the vertices of $N_G(u) - N_G(v) - v$, and if $v \notin S$ then necessarily $S \cap (N_G(u) - N_G(v) - v) \neq \emptyset$, and the only new neighbor acquired by any vertex of $N_G(u) - N_G(v) - v$ is the vertex v. $\square$

Lemma 4.2. *Let $S \subset V(G)$ be such that $v \notin S$, $v \notin N_G(S)$ but $v \in N_{G_{u\to v}}(S)$. Then $u \notin N_{G_{u\to v}}(S)$ and $N_{G_{u\to v}}(S) \cup u \subseteq N_G(S) \cup v$.*

Proof. Since $v \notin N_G(S)$ but $v \in N_{G_{u\to v}}(S)$ we have $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$. Then, given that $v \notin S$, by the previous lemma we must have $S \cap (N_G(u) - N_G(v) - v) \neq \emptyset$ and $N_{G_{u\to v}}(S) \subseteq N_G(S) \cup v$.

Now $S \cap (N_G(u) - N_G(v) - v)$ non-empty implies that $S \cap N_G(u)$ is non-empty, and so $u \in N_G(S)$. On the other hand, we are given that $v \notin N_G(S)$, which implies that $S \cap (N_G(u) \cap N_G(v)) = \emptyset$. Now $N_{G_{u\to v}}(u) \subseteq N_G(u) \cap N_G(v) \cup v$. Since $v \notin S$, and $S \cap (N_G(u) \cap N_G(v)) = \emptyset$, this means we must have $u \notin N_{G_{u\to v}}(S)$.

We have $N_{G_{u\to v}}(S) \subseteq N_G(S) \cup v$, and have just established that $u \in N_G(S)$ and $u \notin N_{G_{u\to v}}(S)$. Thus we have $N_{G_{u\to v}}(S) \cup u \subseteq N_G(S) \cup v$, as required. $\square$

We are now equipped to determine what happens to $N_G(S)$ when u and v are present or not present in S. The next few lemmas are essentially the cases that when taken together produce Theorem 4.7.

Lemma 4.3. *Let G be a graph with $u, v \in V(G)$. Let $S \subseteq V(G)$ be such that $v \notin S$ and $S \cap (N_G(u) - N_G(v) - v) \neq \emptyset$. Then either $N_{G_{u\to v}}(S) \subseteq N_G(S)$, or $u \notin N_{G_{u\to v}}(S), v \notin N_G(S)$ and $N_{G_{u\to v}}(S) \cup u \subseteq N_G(S) \cup v$. In either case, we have $|N_{G_{u\to v}}(S)| \leq |N_G(S)|$.*

Proof. If $N_{G_{u\to v}}(S) \subseteq N_G(S)$ then we are done, so we assume that $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$. Since $v \notin S$, then, by Lemma 4.1 we must have $N_{G_{u\to v}}(S) \subseteq N_G(S) \cup v$. Since we are assuming $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$, then we must have $v \notin N_G(S)$ and $v \in N_{G_{u\to v}}(S)$, and we are given $v \notin S$. By Lemma 4.2 then, we have $u \notin N_{G_{u\to v}}(S)$ and $N_{G_{u\to v}}(S) \cup u \subseteq N_G(S) \cup v$. $\square$

Lemma 4.4. *Let G be a graph with $u, v \in V(G)$. Let $S \subseteq V(G)$ be such that $u, v \in S$. Then $|N_{G_{u\to v}}(S)| \leq |N_G(S)|$.*

Proof. If $N_{G_{u\to v}}(S) \subseteq N_G(S)$ then we are done, so we assume that $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$. Now let $Y = S - \{u, v\}$, so that

$$N_G(S) = N_G(Y) \cup N_G(u,v)$$
$$N_{G_{u\to v}}(S) = N_{G_{u\to v}}(Y) \cup N_{G_{u\to v}}(u,v).$$

After compression $N_G(u,v) = N_{G_{u\to v}}(u,v)$, and since we are assuming that $N_{G_{u\to v}}(S) \nsubseteq N_G(S)$, we must therefore have $N_{G_{u\to v}}(Y) \nsubseteq N_G(Y)$. Since $v \notin Y$ by definition, by Lemma 4.1 we have $Y \cap (N_G(u) - N_G(v) - v) \neq \emptyset$. But then by Lemma 4.3, with Y in place of S, we must have $u \notin N_{G_{u\to v}}(Y), v \notin N_G(Y)$ and $N_{G_{u\to v}}(Y) \cup u \subseteq N_G(Y) \cup v$. Thus

$$N_{G_{u\to v}}(u,v) \cup N_{G_{u\to v}}(Y) \cup u \subseteq N_G(u,v) \cup N_G(Y) \cup v$$

with $u \notin N_{G_{u\to v}}(Y)$ and $v \notin N_G(Y)$. Now if $uv \in E(G)$ then both u and v are in both of the sets $N_G(u,v)$ and $N_{G_{u\to v}}(u,v)$, and the above then implies that $N_{G_{u\to v}}(S) \subseteq N_G(S)$, a contradiction. So we must have $uv \notin E(G)$. In this case then both u and v are absent from both of the sets $N_G(u,v)$ and $N_{G_{u\to v}}(u,v)$. This then implies that

$$N_{G_{u\to v}}(S) \cup u \subseteq N_G(S) \cup v$$

where $u \notin N_{G_{u\to v}}(S)$ and $v \notin N_G(S)$, which gives $|N_G(S)| \leq |N_{G_{u\to v}}(S)|$ as required. $\square$

Lemma 4.5. *Let G be a graph with $u, v \in V(G)$ and $uv \notin E(G)$. Let $S \subseteq V(G)$ be such that $u \notin S$ and $v \in S$. Then $|N_{G_{u\to v}}(S')| \leq |N_G(S)|$, where $S' = S - v \cup u$.*

Proof. Let $Y = S' - u = S - v$ and note that $v \notin Y$. We have

$$N_G(S) = N_G(Y) \cup N_G(v)$$
$$N_{G_{u\to v}}(S') = N_{G_{u\to v}}(Y) \cup N_{G_{u\to v}}(u).$$

Since $uv \notin E(G)$, we have $N_{G_{u\to v}}(u) \subseteq N_G(v)$. Thus if $N_{G_{u\to v}}(Y) \subseteq N_G(Y)$ then $N_{G_{u\to v}}(S') \subseteq N_G(S)$ and we are done. So we may assume that $N_{G_{u\to v}}(Y) \nsubseteq N_G(Y)$. By Lemma 4.2 we have $u \notin N_{G_{u\to v}}(Y), v \notin N_G(Y)$ and $N_{G_{u\to v}}(Y) \cup u \subseteq N_G(Y) \cup v$. But then

$$N_{G_{u\to v}}(S') \cup u \subseteq N_{G_{u\to v}}(Y) \cup N_{G_{u\to v}}(u) \cup u \subseteq N_G(Y) \cup N_G(v) \cup v = N_G(S) \cup v$$

with $u \notin N_{G_{u\to v}}(S'), v \notin N_G(S)$. Hence $|N_{G_{u\to v}}(S')| \leq |N_G(S)|$ in this case as well. $\square$

Lemma 4.6. *Let G be a graph with $u, v \in V(G)$ and $uv \in E(G)$. Let $S \subseteq V(G)$ be such that $u \notin S$ and $v \in S$. Then $|N_{G_{u\to v}}(S')| \leq |N_G(S)|$, where $S' = S - v \cup u$.*

Proof. As in the previous lemma set $Y = S' - u = S - v$. We consider the following two cases.

Case 1. $Y \cap N_G(u) \neq \emptyset$. Then since $u \in N_G(Y)$, and $v \in N_{G_{u\to v}}(Y)$, we have $N_G(S) = N_{G-uv}(S)$ and $N_{G_{u\to v}}(S') = N_{G_{u\to v}-uv}(S')$. Hence we may consider instead the graphs $G - uv$ and $G_{u\to v} - uv$ and the result follows from the previous lemma.

Case 2. $Y \cap N_G(u) = \emptyset$. Again we use the fact that the previous lemma applied to $G - uv$ and $G_{u\to v} - uv$ gives $|N_{G_{u\to v}-uv}(S')| \leq |N_{G-uv}(S)|$. Now in this case $u \notin N_G(Y)$ and hence

$N_G(S) = N_{G-uv}(S) \cup u$ with $u \notin N_{G-uv}(S)$, which implies $|N_G(S)| = |N_{G-uv}| + 1$. But in $G_{u\to v}$, since $u \in S'$ but $v \notin S'$ then removing uv can at most remove v from $N_{G_{u\to v}}(S')$. Thus $N_{G_{u\to v}}(S') \subseteq N_{G_{u\to v}-uv}(S') \cup v$, which implies $|N_{G_{u\to v}}(S')| \le |N_{G_{u\to v}-uv}(S')| + 1$. Thus we have

$$|N_{G_{u\to v}}(S')| \le |N_{G_{u\to v}-uv}(S')| + 1 \le |N_{G-uv}(S)| + 1 \le |N_G(S)|$$

as required. □

Taken together, the previous lemmas imply the following theorem, which analogous to Theorems 2.3 and 3.3.

Theorem 4.7. *Let G be a graph with $u, v \in V(G)$. Then for any fixed k,*

$$\min_{\substack{S'\subseteq V(G_{u\to v}),\\ |S'|=k}} |N_{G_{u\to v}}(S')| \le \min_{\substack{S\subseteq V(G),\\ |S|=k}} |N_G(S)|. \tag{3}$$

Proof. It suffices to show that for any $S \subset V(G)$, there exists an $S' \subset V(G_{u\to v})$ such that

1. $|S'| = |S|$, and
2. $|N_{G_{u\to v}}(S')| \le |N_G(S)|$.

We exhaust all possible cases. If $v \notin S$ and $S \cap (N_G(u) - N_G(v) - v) = \emptyset$ then, as noted in Lemma 4.1, we have $N_{G_{u\to v}}(S) \subseteq N_G(S)$ and we may take $S' = S$. If $v \notin S$ and $S \cap (N_G(u) - N_G(v) - v) \neq \emptyset$, then by Lemma 4.3 we have $|N_{G_{u\to v}}(S)| \le |N_G(S)|$ and we may take $S' = S$. So now we consider $v \in S$. If $u, v \in S$ then Lemma 4.4 gives $|N_{G_{u\to v}}(S)| \le |N_G(S)|$, and again we may take $S' = S$. And finally, if $u \notin S$ but $v \in S$, Lemmas 4.5 and 4.6 give $|N_{G_{u\to v}}(S')| \le |N_G(S)|$ for $S' = S - v \cup u$, which satisfies $|S'| = |S - v \cup u| = |S|$ since $v \in S$ but $u \notin S$. Thus the theorem statement holds for either the S' of Lemmas 4.5 and 4.6 or for $S' = S$ itself. □

Finally, as a straightforward algebraic consequence of the previous theorem we have the result on binding number.

Theorem 4.8. *Let G be a graph with $u, v \in V(G)$. Then* $\mathrm{bind}(G_{u\to v}) \le \mathrm{bind}(G)$.

5 Compression and Threshold Graphs

Threshold graphs are a well-known and much studied class of graphs and there are many equivalent ways to define them, see for example [22]. For our purposes, one of the more informative definitions involves a dominance relation on vertices. We say *vertex v dominates vertex u in graph G* if $N_G(u) \subseteq N_G[v]$, where $N_G[v]$ is the closed neighborhood $N_G[v] = N_G(v) \cup v$. A *threshold graph* is a graph in which, given any pair of vertices $u, v \in V(G)$, either u dominates v or v dominates u.

If $G_{u\to v} \neq G$ then the compression operation takes two vertices $u, v \in V(G)$ which do not dominate each other and produces a new graph $G_{u\to v}$ in which v does dominate u. After compression, then, a graph is "more threshold" and continued application of compression can only increase this. Eventually, continuing the compression operation (with different pairs of vertices) will result in a threshold graph, a fact noted originally by Satyanarayana, Schoppmann, and Suffel [23], as well as by a number of authors since then [4, 11, 13, 14].

Theorem 5.1. *Let G_0 be a connected graph. Then there exists a series of connected graphs $G_0, G_1, G_2, \ldots, G_k$ such that, for all $i = 1, \ldots, k$, the graph G_i is connected with $G_i = (G_{i-1})_{u \to v}$ for some $u, v \in V(G_{i-1})$, and the graph G_k is a connected threshold graph.*

Since compression decreases spanning trees and all-terminal reliability, it can be concluded [7, 23] that for any connected graph G there is a connected threshold graph H such that $\tau(H) \leq \tau(G)$ and $Rel_p(H) \leq Rel_p(G)$. In other words, threshold graphs minimize those parameters. In the same way, Theorem 5.1 and the results of this paper imply the following.

Theorem 5.2. *Let G be a connected graph. Then there exists a connected threshold graph H with the same number of vertices and edges such that $p(H) \leq p(G)$, where p is any of the vulnerability parameters connectivity, toughness, edge connectivity, edge toughness, and binding number. In other words, threshold graphs minimize those parameters.*

Theorem 5.3. *Let G be a connected graph. Then there exists a connected threshold graph H with the same number of vertices and edges such that $sc(H) \geq sc(G)$, where sc denotes the scattering number. In other words, threshold graphs maximize scattering number.*

An interesting question is which particular threshold graphs minimize (or maximize) those parameters. Another useful way of defining threshold graphs are as particular kinds of *split graphs*, graphs whose vertex sets can be partitioned into two sets, one of which induces a clique and the other of which induces an independent set. A threshold graph is a split graph in which the vertices of the independent set have nested neighborhoods [22]. In 1990 Boesch et al. conjectured [2] the form of the particular threshold graph $L_{n,m}$ that would, for given $n = |V(G)|$ and $m = |E(G)|$, achieve the minimum values of $\tau(G)$ and $Rel_p(G)$. Informally, these are obtained by making the clique as large as possible and then, when connecting the remaining vertices of the independent set to the clique, making as many degree one vertices as possible. Formally, let k be the least integer such that $m \geq \binom{n-k}{2} + k$. Then $L_{n,m}$ is the threshold graph consisting of an $(n-k)$-clique, with $k-1$ pendant vertices and one vertex of degree $m - \binom{n-k}{2} - k - 1$ attached to it. Confirming Boesch's conjecture that $L_{n,m}$ minimizes $\tau(G)$ and $Rel_p(G)$ for all p appears to be difficult. It took about 20 years for the conjecture to be proven correct for $\tau(G)$ in [4], with the proof requiring a long and technical optimization argument. The conjecture that the $L_{n,m}$ graphs minimize all-terminal reliability remains open.

For some of the parameters considered here the situation is not so difficult. For instance, it is clear that minimizing κ and λ for threshold graphs can be achieved by creating a pendant vertex or, when that is not possible, by minimizing the minimium degree δ, and it is also clear that the $L_{n,m}$ graphs do this. Hence the $L_{n,m}$ graphs minimize κ and λ. For the other parameters, we do not know the minimizing (or in the case of scattering number, maximizing) graphs.

Question 1. *Which graphs minimize toughness, edge toughness, and binding number, and maximize scattering number, over all connected graphs with n vertices and m edges?*

The $L_{n,m}$ graphs are typically not unique minimizers, even for spanning trees; for instance the $k-1$ pendant edges can be replaced with a path on those vertices instead and the resulting graph will have the same number of vertices, edges, and spanning trees as the equivalent $L_{n,m}$ graph. In [3], however, Bogdanovich showed that compression preserves the property of being 2-connected chordal, and furthermore proved in this case that repeated application of compression results in a unique 2-connected chordal threshold graph that minimizes spanning trees over all 2-connected chordal graphs, which we describe here. Let k be the least integer such that $m \geq \binom{n-k}{2} + 2k$. Then $Q_{n,m}$ is the unique threshold graph consisting of an $(n-k)$-clique, and $k-1$ vertices of degree 2 and one vertex of degree $m - \binom{n-k}{2} - 2k - 1$ attached to it. Therefore, as with Theorem 5.2, we have the following.

Theorem 5.4. *Let G be a 2-connected chordal graph on n vertices and m edges. Then $p(Q_{n,m}) \leq p(G)$, where p is any of toughness, edge toughness, and binding number, and $sc(Q_{n,m}) \geq sc(G)$ for scattering number. In other words, $Q_{n,m}$ minimizes toughness, edge toughness, and binding number, and maximizes scattering number, over all 2-connected chordal graphs.*

We conclude with a final comment on graph vulnerability parameters and compression. The fact that threshold graphs have been shown to be extremal for all vulnerability parameters to date suggests the "meta-conjecture" that perhaps compression always weakens a graph, or that the $L_{n,m}$ threshold graphs might be extremal graphs for any vulnerability parameter. These conjectures are in fact not true. In a companion paper [18] to his one we consider the vulnerability parameters integrity, tenacity, and k-order connectivity under compression and show that there are graphs containing a pair of vertices u, v where compression from u to v can increase those parameters. In other words, and in contrast to the parameters examined in this paper, when "strength" is measured by those parameters compression may strengthen certain graphs. It is also shown that the $L_{n,m}$ graphs are not minimizers for those parameters for certain n, m values. However in that paper it is also shown that the larger class of quasi-threshold graphs minimize the integrity, tenacity, and k-ordered connectivity parameters. Since threshold graphs are a subclass of quasi-threshold graphs, it is still an open question—and some small examples of this phenomenon are presented—whether threshold graphs other than the $L_{n,m}$ graphs are the minimizers there.

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References

[1] C.A. Barefoot, R. Entringer, H. Swart, Vulnerability in graphs—a comparative survey. *J. Comb. Math. Comb. Comput.* 1 (1987), 12—22.

[2] F.T. Boesch, A. Satyanarayana, C.L. Suffel, Least reliable networks and reliability domination. *IEEE Trans. Commun.* 38 (1990) 2004–2009.

[3] Z. Bogdanowicz, Chordal 2-connected graphs and spanning trees. *J. Graph Theory* 76 (2014), 224–235.

[4] Z. Bogdanowicz, Undirected simple connected graphs with minimum number of spanning trees. *Disc. Math.* 309 (2009), 3074–3082.

[5] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 2002.

[6] H. Broersma, J. Fiala, P.A. Golovach, T. Kaiser, D. Paulusma, A. Proskurowski. Linear-Time Algorithms for Scattering Number and Hamilton-Connectivity of Interval Graphs. *J. Graph Theory* 79 (2015), 282–299.

[7] J. Brown, C. Colbourn and J. Devitt, Network transformations and bounding network reliability. *Networks* 23 (1993), 1–17.

[8] V. Chvátal, On Hamilton's ideals. *J. Combin. Theory Ser. B* 12 (1972), 163–168.

[9] M.B. Cozzens, D. Moazzami, S. Stueckle, The tenacity of the Harary graphs, *J. Combin. Math. Combin. Comput.* 16 (1994), 33–56.

[10] M.B. Cozzens, D. Moazzami,S. Stueckle, The Tenacity of a Graph, in: Yousef Alavi, Allen Schwenk (Eds.), *Graph Theory, Combinatorics, and Algorithms*, Wiley, New York, 1995, pp. 1111-–1122.

[11] P. Csikvári, Applications of the Kelmans transformation: extremality of threshold graphs. *Electron. J. Combin.* 18 (2011), Paper 182, 24 pp.

[12] P. Csikvári, On a conjecture of V. Nikiforov. *Disc. Math.* 309 (2009), 4522–4526.

[13] J. Cutler, A.J. Radcliffe, Extremal graphs for homomorphisms. *J. Graph Theory* 67 (2011), 261–284.

[14] J. Cutler, A.J. Radcliffe, Extremal graphs for homomorphisms II. *J. Graph Theory* 76 (2014), 42–59.

[15] D. Gross, N. Kahl, and J.T. Saccoman, Graphs with the maximum or minimum number of k-Factors. *Disc. Math.* 310 (2010), 687–691.

[16] D. Gusfield, Connectivity and edge-disjoint spanning trees. *Inform. Process. Lett.* 16 (1983), 87–89.

[17] H.A. Jung, On a class of posets and the corresponding comparability graphs. *J. Combinatorial Theory Ser. B* 24 (1978), 125–133.

[18] N. Kahl, Graph vulnerability parameters, compression, and quasi-threshold graphs. *Discrete Appl. Math.* 259 (2019), 119–126.

[19] A.K. Kelmans, On graphs with randomly deleted edges. *Acta. Math. Acad. Sci. Hung.* 37 (1981), 77–88.

[20] A.K. Kelmans, Operations on graphs increasing some graph parameters. arXiv:1109.4622 (2013)

[21] L. Keough, A.J. Radcliffe, Graphs with the fewest matchings. *Combinatorica* 36 (2016), 703–723.

[22] N.V.R. Mahadev and U.N. Peled. Threshold Graphs and Related Topics. *Annals of Discrete Math. Vol. 56*, North-Holland Publishing Co., Amsterdam, 1995.

[23] A. Satyanarayana, L. Schoppmann, C. Suffel, A reliability-improving graph transformation with applications to network reliability. *Networks* 22 (1992), 209–216.

[24] D.R. Woodall. The binding number of a graph and its Anderson number. *J. Combin. Theory Ser. B* 15 (1973), 225–255.