Extremal Graphs for the Tutte Polynomial

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Abstract

A graph transformation called the compression of a graph G is known to decrease the number of spanning trees, the all-terminal reliability, and the magnitude of the coefficients of the chromatic polynomial of a graph G. All of these graph parameters can be derived from the Tutte polynomial of G, and in this paper we determine more generally compression's effect on the Tutte polynomial, recovering the previous results and obtaining similar results for a wide variety of other graph parameters derived from the Tutte polynomial. Since any simple connected graph can be transformed into a connected threshold graph via a series of compressions, this gives that threshold graphs are extremal simple graphs for all of the parameters considered.

1 Introduction

The Tutte polynomial $T_G(x, y)$ of a graph G with n vertices may be defined as

$$T_G(x, y) = \sum_{A \subseteq E(G)} (x-1)^{\kappa(A)-\kappa(E(G))}(y-1)^{|A|-n+\kappa(A)}$$

where $\kappa(A)$ is the number of connected components of the spanning subgraph defined by the edges of A. The polynomial $T_G(x, y)$ has been shown to encode a vast amount of structural information about G, in essence encompassing every graph parameter of G that obeys a deletion-contraction identity, a property sometimes called the universality property of the Tutte polynomial [20]. We mention three parameters in particular:

- the number of spanning trees of a connected graph G, denoted $\tau(G)$, is $T_G(1, 1)$,
- the all-terminal reliability of G, denoted $Rel_G(p)$, gives the probability that G is connected if the edges of G fail independently with probability p, and is

$$Rel_G(p) = p^{|V(G)|-1}(1-p)^{|E(G)|-|V(G)|+1}T_G(1, 1/(1-p)),$$

- the chromatic polynomial of G, denoted $\chi_G(\lambda)$, gives the number of ways to properly color the vertices of G with λ colors, and is

$$\chi_G(\lambda) = (-1)^{|V|-\kappa(E(G))}\lambda^{\kappa(E(G))}T_G(1-\lambda, 0).$$

For these facts and more on the Tutte polynomial and its applications we refer to the surveys [10, 20, 35, 45].

Threshold graphs are a well-known and much-studied class of graphs, see for instance [32]. In Section 6 of this paper we discuss them in more detail; for now we observe that threshold graphs have been shown to minimize, over all simple connected graphs, the three parameters listed above. In his 1985 thesis [5] Bogdanowicz showed that for any connected graph G there was a connected threshold graph H with the same number of vertices and edges such that $\tau(H) \leq \tau(G)$, and subsequently Satyanarayana, Schoppmann, and Suffel [38] rediscovered the spanning tree result and showed a similar statement held for the all-terminal reliability $Rel_G(p)$. Letting $c_i(G)$ denote the i^{th} coefficient of $\chi_G(\lambda)$, in [14] Csikvári and in [37] Rodriguez and Satyanarayana showed that for every G there is a threshold graph H with the same number of vertices and edges such that $|c_i(H)| \leq |c_i(G)|$ for every $i \geq 1$.

Given these facts a few questions arise. Might threshold graphs minimize a larger collection of graph parameters related to the Tutte polynomial? Do threshold graphs maximize any such parameters? Are threshold graphs even in some sense extremal for the Tutte polynomial itself?

In this paper we show that in a very natural sense threshold graphs are minimal for the Tutte polynomial, and that in a unified way this minimality translates into minimality (or in rare cases, maximality) for a wide variety of graph parameters that can be derived from it. It is probably worth mentioning here some of these parameters, which for convenience we categorize as either evaluations of the Tutte polynomial or polynomial specializations of it. For evaluations, we are able to show that threshold graphs are extremal for every evaluation of $T_G(x, y)$ in the first quadrant of the xy-plane, and can characterize for which points there threshold graphs are minimal or maximal. The number of spanning trees and the all-terminal reliability are two such evaluations, but others here include the number of spanning forests and the number of spanning connected subgraphs; the number of acyclic orientations, totally cyclic orientations, acyclic orientations with a single source, or score vectors of orientations; enumerations of a wide variety of different types partial orientations; and the evaluations of the q-state Potts model from theoretical physics for $q \geq 1$. (Threshold graphs minimize all of these parameters except the q-state Potts model evaluations, which are maximized.) Looking at polynomial specializations of $T_G(x, y)$, in addition to recovering the aforementioned chromatic polynomial result, we can for example derive similar coefficient results for the flow polynomial, the all-terminal reliability polynomial (in both its S-form and H-form) and various generating functions, including generating functions for spanning forests of i components, spanning subgraphs of i edges, the number of critical configurations of level i of the Abelian sandpile model, and variety of generating functions relating to fourientations of a graph.

The main tool used to accomplish this is a graph transformation called the *compression of G from vertex u to vertex v*. Let $N_G(u)$ and $N_G(v)$ respectively denote the neighborhoods in G of u and v. The compression of G from u to v produces a new graph $G_{u\to v}$ by, for each $x \in N_G(u) - N_G(v) - \{v\}$, removing all edges from G of the form ux and replacing them with corresponding edges of the form vx. An illustration of compression appears in Figure 1. Graph compression appears to have been first employed by Kelmans in [28], who to a large extent anticipated the spanning tree and all-terminal reliability results of [5, 38] by showing that compression decreased those parameters. Graph compression also has been shown to uniformly decrease or increase a number of graph parameters not associated with the Tutte polynomial [15, 16, 17, 24, 25, 26, 29]. We note that the operation does not have a standard name; in the literature we have also seen it called a path property transformation [5], a shift transformation [6, 7, 9], a swing surgery [38], a

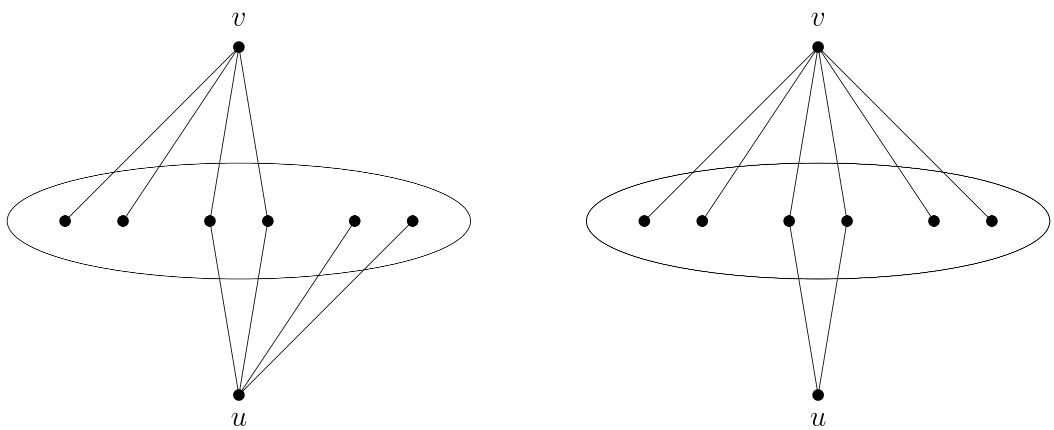

Figure 1: An illustration of compression. On the left is a graph G, and on the right $G_{u\to v}$, the compression of G from u to v. If edge uv had been present in G, it would also be present in $G_{u\to v}$.

Kelmans transformation [14, 15], or not given a name at all (as in Kelmans' original paper). Our choice of the "compression" terminology here follows [16, 17] and is motivated by the following fact—first observed by Bogdanowicz [5] and by Satyanarayana, Schoppmann, and Suffel [38], but also by a number of authors since [6, 14, 16, 17]—which gives it its relevance for extremality. (Connected threshold graphs necessarily have diameter less than or equal to 2.)

Theorem 1.1 ([5, 38])**.** *Any connected non-threshold graph G can be transformed into a threshold graph H by repeated applications of graph compression. We may also take the threshold graph H to be connected by taking each compression to be on vertices at distance 2 or less.*

The main theorem of this paper determines the effects of the compression of G from u to v upon $T_G(x,y)$, showing that in an important respect compression "decreases" the Tutte polynomial. Letting $dist_G(u,v)$ denote the distance in G from u to v, we prove the following.

Theorem 1.2. *Let G be a connected graph, and let $u,v \in V(G)$. If $dist_G(u,v) \leq 2$ then*

$$T_G(x,y) - T_{G_{u\to v}}(x,y) = (x+y-xy)P_{u,v}(x,y)$$

and if $dist_G(u,v) \geq 3$ then

$$T_G(x,y) - (x-1)T_{G_{u\to v}}(x,y) = (x+y-xy)P_{u,v}(x,y)$$

where $P_{u,v}(x,y)$ is a polynomial with non-negative coefficients.

In this paper we will primarily be concerned with connected graphs and will mostly focus on the $dist_G(u,v) \leq 2$ case above (compression when $dist_G(u,v) \geq 3$ renders u an isolate) although the $dist_G(u,v) \geq 3$ case will be be useful too, for example helping to provide a simple algebraic proof of Csikvári's results on compression's effect on the chromatic polynomial.

It is not difficult to see that the subtraction of the Tutte polynomials of two connected graphs with the same number of vertices and edges must have a factor of $x+y-xy$, a fact we demonstrate later in Section 5. The key feature of Theorem 1.2 then is that, regardless of the choice of $u,v \in$

$V(G)$, the polynomials $P_{u,v}(x, y)$ have non-negative coefficients. In this sense $T_{G_{u\to v}}(x, y)$ is a "smaller" polynomial than $T_G(x, y)$, and it is this decrease that carries over to the array of graph parameters determined by the Tutte polynomial previously mentioned. That threshold graphs are typically minimizing for these parameters follows from these decreases and Theorem 1.1.

The structure of the paper is as follows. We approach the proof of Theorem 1.2 via the *multivariate q-state Potts model* $Z_G(q, \mathbf{w})$, also called the *multivariate Tutte polynomial* [41]. This polynomial generalizes the q-state Potts model $Z_G(q, w)$ of theoretical physics, which in turn is equivalent to the traditional Tutte polynomial via a change of variables. Our proof of Theorem 1.2 is fairly computational and the multivariate q-state Potts model has a number of computational advantages over the traditional Tutte polynomial, including a simpler deletion-contraction rule and series and parallel reductions. Our use of the Potts model, for example, permits a unified approach to the $dist_G(u, v) \leq 2$ and $dist_G(u, v) \geq 3$ cases and also helps explain the appearance of the $x - 1$ factor in the $dist_G(u, v) \geq 3$ case. In order to keep the paper as self-contained as possible, in Section 2 we present the necessary preliminaries on the multivariate q-state Potts model. In Section 3 we prove an analogue to Theorem 1.2 for the q-state Potts model, which serves both as the base of our proof of Theorem 1.2 and as a simpler model of it. In Section 4 we adapt the results of Section 3 to the traditional Tutte polynomial setting, proving Theorem 1.2.

In Section 5 we introduce the (n, m) *Tutte polynomial poset*, a poset on simple connected graphs of n vertices and m edges where $H \preccurlyeq G$ if and only if

$$T_G(x, y) - T_H(x, y) = (x + y - xy)P(x, y)$$

for some polynomial $P(x, y)$ with non-negative coefficients. We show how $H \preccurlyeq G$ in this poset implies a decrease in all of the graph parameters mentioned previously (except the evaluations of the q-state Potts model itself, for which $H \preccurlyeq G$ implies an increase), allowing us to address all of these in a unified way. The first case of Theorem 1.2 demonstrates that $G_{u\to v} \preccurlyeq G$ when $dist_G(u, v) \leq 2$ and thus we have that these compressions decrease (or increase) all of the relevant parameters. Finally in Section 6 we conclude with a discussion of threshold graphs in the context of the (n, m) Tutte polynomial posets, confirming that they are minimal there: for any simple connected graph G there is a connected threshold graph H with the same number of vertices and edges such that $H \preccurlyeq G$. We end by conjecturing that the (n, m) Tutte polynomial posets in fact have a minimum element, in other words that there exists a threshold graph $L_{n,m}$ such that $L_{n,m} \preccurlyeq G$ for every G with n vertices and m edges, and we describe these threshold graphs $L_{n,m}$ which we believe are minimum graphs for the Tutte polynomial.

2 The multivariate q-state Potts model

The multivariate q-state Potts model polynomial is defined to be

$$Z_G(q, \mathbf{w}) = \sum_{A \subseteq E} q^{\kappa(A)} \prod_{e \in A} w_e$$

where q and the elements of $\mathbf{w} = \{w_e\}_{e\in E}$ are commuting indeterminates. When the w_e are all equal, say $w_e = w$ for all $e \in E(G)$, the resulting polynomial is the q-state Potts model from theoretical physics, denoted $Z_G(q, w)$. We now present the facts about $Z_G(q, \mathbf{w})$ and $Z_G(q, w)$ necessary for our main results. All of what follows here up until Theorem 2.1 can be found elsewhere, for example in the excellent survey by Sokal [41].

Unlike the traditional Tutte polynomial $T_G(x, y)$ the deletion-contaction rule for $Z_G(q, \mathbf{w})$ is the same for any edge e, regardless of whether e is a bridge, loop, or otherwise:

$$Z_G(q, \mathbf{w}) = Z_{G-e}(q, \mathbf{w}) + w_e Z_{G/e}(q, \mathbf{w}) \tag{1}$$

where $G - e$ indicates G with the edge e deleted, and G/e indicates G with the edge e contracted. Any multiple edges and/or loops created by contracting an edge are kept in G/e. (On the right-hand side since edge e is no longer present in G we may if desired omit w_e from $\mathbf{w}$.) The deletion-contraction formula (1) can be used as a recursive definition of $Z_G(q, \mathbf{w})$ by applying it successively to all edges present, with the empty graph $\overline{K_n}$ having $Z_{\overline{K^n}}(q, \mathbf{w}) = q^n$. As with the Tutte polynomial the resulting polynomial $Z_G(q, \mathbf{w})$ is independent of the order in which edges are taken.

In the q-state Potts model setting if e is a loop then we take $G/e = G - e$. As a consequence when e is a loop we have

$$Z_G(q, \mathbf{w}) = (1 + w_e) Z_{G-e}(q, \mathbf{w}). \tag{2}$$

Like the traditional Tutte polynomial $T_G(x, y)$, the multivariate Tutte polynomial factors over components; if G is the disjoint union of graphs G_1 and G_2 then

$$Z_G(q, \mathbf{w}) = Z_{G_1}(q, \mathbf{w}) Z_{G_2}(q, \mathbf{w}). \tag{3}$$

In particular, adding an isolated vertex to a graph G has the effect of multiplying $Z_G(q, \mathbf{w})$ by q, i.e. $Z_{G \cup K_1}(q, \mathbf{w}) = q Z_G(q, \mathbf{w})$. In a similar manner, if G consists of graphs G_1 and G_2 joined at a single cutvertex, then

$$Z_G(q, \mathbf{w}) = \frac{1}{q} Z_{G_1}(q, \mathbf{w}) Z_{G_2}(q, \mathbf{w}). \tag{4}$$

As a consequence when e is a bridge we have $Z_{G-e}(q, \mathbf{w}) = q Z_{G/e}(q, \mathbf{w})$, and then by deletion-contraction on e we have

$$Z_G(q, \mathbf{w}) = (q + w_e) Z_{G/e}(q, \mathbf{w}). \tag{5}$$

This implies that all trees with constant weights on n vertices have the same q-state Potts model polynomial $q(q + w)^{n-1}$.

Unlike the traditional Tutte polynomial, the multivariate q-state Potts model permits reductions of edges in series or in parallel. We require only the parallel reduction. A set of edges $e_1, \ldots, e_k$ are said to be *in parallel* if they share the same endvertices. If edges $\{e_i\}_{i=1}^k$ are in parallel then the *parallel reduction* replaces all of the edges by a single edge e with weight

$$w_e = \prod_{i=1}^{k} (w_{e_i} + 1) - 1. \tag{6}$$

The multivariate q-state Potts model is invariant under parallel reductions, i.e., $Z_G(q, \mathbf{w}) = Z_{G'}(q, \mathbf{w}')$ where G' and $\mathbf{w}'$ are the graph G and the weight vector $\mathbf{w}$ after the parallel reduction. We may allow edge weights of zero to indicate the absence of edges between two vertices, which is reflected in the form of (6) as well.

The proofs of our main results essentially proceed by reducing the problem to a particular bipartite case. The bipartite graph of interest here we will call a *multigraph* $K_{2,p}$, that is, the bipartite graph $K_{2,p}$ where various edges may appear with some multiplicity (including possibly multiplicity of zero). The *generalized theta graph* $\theta_{s_1,\ldots,s_p}$ is the graph consisting of endvertices u, v connected by p internally disjoint paths of lengths $s_1, \ldots, s_p$. In [40] Sokal determined $Z_G(q, \mathbf{w})$ for G a generalized theta graph.

Theorem 2.1 (Proposition 2.3, [40]). *For the generalized theta graph $\theta_{s_1,\ldots,s_p}$ with edge weights $\{w_{ij}\}_{1\le i\le p, 1\le j\le s_i}$ we have*

$$\begin{aligned} &Z_{\theta_{s_1,\ldots,s_p}}(q,\{w_{ij}\}) \\ &= q^{-(p-1)}\left\{\prod_{i=1}^{p}\left[\prod_{j=1}^{s_i}(q+w_{ij})+(q-1)\prod_{j=1}^{s_i}w_{ij}\right]+(q-1)\prod_{i=1}^{p}\left[\prod_{j=1}^{s_i}(q+w_{ij})-\prod_{j=1}^{s_i}w_{ij}\right]\right\}. \end{aligned} \tag{7}$$

Using the generalized theta graph result above with the parallel reduction we obtain $Z_G(q,w)$ for G a multigraph $K_{2,p}$.

Lemma 2.2. *Let G be a multigraph $K_{2,p}$, with $\{u,v\}$ one partite set and $\{x_i\}_{i=1}^p$ the other, and for each $i=1,\ldots,p$ let m_i (respectively n_i) denote the number of parallel edges between x_i and u (respectively v). Then*

$$Z_G(q,w)=q\left[\prod_{i=1}^{p}\left(q+(w+1)^{m_i+n_i}-1\right)+(q-1)\prod_{i=1}^{p}\left(q+(w+1)^{m_i}+(w+1)^{n_i}-2\right)\right].$$

Proof. The simple graph $K_{2,p}$ is a generalized theta graph with $s_i=2$ for all $i=1,\ldots,p$. Now (7) gives

$$\begin{aligned} Z_{K_{2,p}}(q,\mathbf{w}) &= q^{-(p-1)}\left[\prod_{i=1}^{p}((q+w_{i1})(q+w_{i2})+(q-1)w_{i1}w_{i2})\right. \\ &\qquad\left.+(q-1)\prod_{i=1}^{p}((q+w_{i1})(q+w_{i2})-w_{i1}w_{i2})\right] \\ &= q^{-(p-1)}\left[\prod_{i=1}^{p}(q^2+qw_{i1}+qw_{i2}+qw_{i1}w_{i2})+(q-1)\prod_{i=1}^{p}(q^2+qw_{i1}+qw_{i2})\right] \end{aligned}$$

where $\mathbf{w}=\bigcup_{i=1}^{p}\{w_{i1},w_{i2}\}$. Factoring q^p from the brackets and rewriting the first product we obtain

$$Z_{K_{2,p}}(q,\mathbf{w})=q\left[\prod_{i=1}^{p}(q+(w_{i1}+1)(w_{i2}+1)-1)+(q-1)\prod_{i=1}^{p}(q+w_{i1}+w_{i2})\right].$$

When $K_{2,p}$ is a multigraph, the edge weights are given by using the parallel reduction (6) on the parallel edges present. For identical edge weights with weight w we obtain

$$\begin{aligned} w_{i1} &= (w+1)^{m_i}-1 \\ w_{i2} &= (w+1)^{n_i}-1 \end{aligned}$$

for each $i=1,\ldots,p$. If a single edge or no edges are present between two vertices, the parallel reduction formula recovers w or 0 appropriately, hence there is no problem applying the parallel reduction for all $i=1,\ldots,p$. Substituting these for the weights of $\mathbf{w}$ and simplifying gives the result. □

3 Compression and the q-state Potts model

Before proving a q-state Potts model version of Theorem 1.2 we make some comments about notation, applicable here and in the Tutte polynomial proof in the next section. Since $G - u - v = G_{u\to v} - u - v$ there is an obvious and natural correspondence between the edges of those subgraphs and, abusing notation somewhat, we will use the same labels to identify the vertices and edges in $G - u - v$ and the corresponding vertices and edges in $G_{u\to v} - u - v$. We will also use the following notation, first appearing in [16, 17]. The choice of $u, v \in V(G)$ defines a natural partition of $V(G) - \{u, v\}$ into four parts: vertices adjacent only to u, vertices adjacent only to v, vertices adjacent to both, and vertices adjacent to neither. For ease of reference we abbreviate these as $A_{u\overline{v}}$, $A_{\overline{u}v}$, A_{uv}, and $A_{\overline{uv}}$ respectively. Finally, for brevity we may refer to a polynomial with all non-negative coefficients as a *non-negative polynomial*, and a non-negative polynomial with at least one positive coefficient, i.e. one that is not the zero polynomial, as a *positive polynomial*.

Theorem 3.1. *Let G be a connected graph, and let $u, v \in V(G)$. Then*

$$Z_G(q, w) - Z_{G_{u\to v}}(q, w) = q(1 - q)P_{u,v}(q, w)$$

where $P_{u,v}(q, w)$ is a non-negative polynomial.

Proof. To minimize the appearance of multiple subscripts during calculations throughout the proof we will let $G_{u\to v} = H$.

If edge $uv \in E(G)$ then, noting that $G/uv = H/uv$, by deletion-contraction on uv we have

$$Z_G(q, w) - Z_H(q, w) = Z_{G-uv}(q, w) - Z_{H-uv}(q, w).$$

Hence we may assume that there is no edge between u and v. Additionally, if $E(G - u - v)$ is empty then G and H have the same block decomposition and $Z_G(q, v) = Z_H(q, v)$. In this case the theorem is true with $P_{u,v}(q, v) = 0$, so we also assume that $E(G - u - v)$ is non-empty.

Let $S = E(G - u - v) = E(G_{u\to v} - u - v)$. We now decompose $Z_G(q, w)$ and $Z_H(q, w)$ by, in both graphs, "deleting/contracting every edge of S." Formally, let $\{e_1, \ldots, e_\ell\}$ be a fixed but arbitrary ordering of the edges of S, and let $R = (r_1, \ldots, r_\ell)$ be a binary vector in $\{0, 1\}^\ell$. Form the multigraphs G_R and H_R by sequentially deleting when $r_i = 0$, or contracting when $r_i = 1$, each edge e_i in turn. Repeated application of the deletion-contraction formula (1) to $Z_G(q, w)$ and $Z_H(q, w)$ then produces

$$Z_G(q, w) - Z_H(q, w) = \sum_{R\in\{0,1\}^\ell} w^r (Z_{G_R}(q, w) - Z_{H_R}(q, w))$$

where $r = \sum_{i=1}^{\ell} r_i$ is the number of contractions that took place under R. Since all edges of S have been deleted or contracted, and the edge uv is not present, then for each R the graphs G_R and H_R are multigraph $K_{2,p}$'s, where $p = |V(G)| - r - 2$.

As in Lemma 2.2, then, for any fixed $R \in \{0, 1\}^\ell$ we let $\{x_i\}_{i=1}^p$ be the non-u, v partite set of G_R and H_R, and for each $i = 1, \ldots, p$ we let m_i (respectively n_i) denote the number of edges between x_i and u (respectively v) in G_R, and define m'_i and n'_i similarly for H_R. (We note that, as with the variable r, each $\{x_i\}_{i=1}^p$ set and the p, m_i, n_i, m'_i, n'_i variables are a function of R, although we omit any identifying R notation for clarity in the calculations to follow.) Since compression preserves the degrees of non-u, v vertices we have

$$m_i + n_i = deg_{G_R}(x_i) = deg_{H_R}(x_i) = m'_i + n'_i.$$

In particular we have
$$q + (w+1)^{m_i+n_i} - 1 = q + (w+1)^{m_i'+n_i'} - 1$$
for each $i = 1, \ldots, p$, and therefore by Lemma 2.2,

$$\begin{aligned} &Z_{G_R}(q,w) - Z_{H_R}(q,w) \\ &\quad = q(1-q)\left[\prod_{i=1}^{p}\left(q + (w+1)^{m_i'} + (w+1)^{n_i'} - 2\right) - \prod_{i=1}^{p}\left(q + (w+1)^{m_i} + (w+1)^{n_i} - 2\right)\right]. \end{aligned} \tag{8}$$

To complete the proof it is now sufficient to show that the expression in brackets above is a polynomial with non-negative coefficients. To that end, let

$$S_{m_i,n_i}(q,w) = q + (w+1)^{m_i} + (w+1)^{n_i} - 2$$

and note that, since m_i, n_i, m_i', n_i' are all non-negative, $S_{m_i',n_i'}(q,w)$ and $S_{m_i,,n_i,}(q,w)$ are themselves polynomials with non-negative coefficients. By adding and subtracting identical terms and then regrouping, we rewrite the bracketed expression in (8) as

$$\prod_{i=1}^{p} S_{m_i',n_i'}(q,w) - \prod_{i=1}^{p} S_{m_i,n_i}(q,w) = \sum_{i=1}^{p}\left[\left(S_{m_i',n_i'}(q,w) - S_{m_i,n_i}(q,w)\right)\prod_{j<i} S_{m_j',n_j'}(q,w)\prod_{j>i} S_{m_j,n_j}(q,w)\right].$$

Since each polynomial $S_{m_i',n_i'}(q,w)$ and $S_{m_i,n_i}(q,w)$ is non-negative, to complete the proof it only remains to show that the expression $S_{m_i',n_i'}(q,w) - S_{m_i,n_i}(q,w)$ is a non-negative polynomial for any single index i.

For a single index i we have

$$\begin{aligned} S_{m_i',n_i'}(q,w) - S_{m_i,n_i}(q,w) &= (w+1)^{m_i'} + (w+1)^{n_i'} - (w+1)^{m_i} - (w+1)^{n_i} \\ &= (w+1)^{m_i'}\left((w+1)^{n_i-m_i'} - 1\right)\left((w+1)^{m_i-m_i'} - 1\right). \end{aligned} \tag{9}$$

We now show that each exponent above is non-negative. Each edge counted by m_i, n_i, m_i', n_i' corresponds, in the original graph G, to an edge from the vertex set $\{u, v\}$ to one of the three sets $A_{u\overline{v}}$, $A_{\overline{u}v}$, or A_{uv}. We now refine those counts by tracking which edges come from each of those three sets. Let $z \rightsquigarrow x_i$ indicate that the vertex z in $V(G)$ (and hence $V(H)$) has been identified with the vertex x_i in G_R (and hence H_R) via the sequence of deletions and contractions in R. For each x_i we define the quantities a_i, b_i, c_i to be

$$\begin{aligned} a_i &= |\{z \in V(G) :\, z \rightsquigarrow x_i \text{ and } z \in A_{u\overline{v}}\}| \\ b_i &= |\{z \in V(G) :\, z \rightsquigarrow x_i \text{ and } z \in A_{\overline{u}v}\}| \\ c_i &= |\{z \in V(G) :\, z \rightsquigarrow x_i \text{ and } z \in A_{uv}\}|. \end{aligned}$$

By the compression operation we have for each x_i,

$$\begin{aligned} m_i &= a_i + c_i & \qquad m_i' &= c_i \\ n_i &= b_i + c_i & \qquad n_i' &= a_i + b_i + c_i \end{aligned}$$

Substituting these a_i, b_i, c_i expressions in (9) gives

$$S_{m'_i,n'_i}(q,w) - S_{m_i,n_i}(q,w) = (w+1)^{c_i}\left((w+1)^{a_i} - 1\right)\left((w+1)^{b_i} - 1\right).$$

Since a_i, b_i, c_i are all non-negative, the expression above is indeed a polynomial with non-negative coefficients, and the proof is complete. $\square$

Theorem 3.1 is used in the proof of our main theorem, and before moving on to that proof it will be useful to note more explicitly the form of $P_{u,v}(q,w)$. From the previous proof we have that

$$Z_G(q,w) - Z_H(q,w) = \sum_{R\in\{0,1\}^\ell} w^r (Z_{G_R}(q,w) - Z_{H_R}(q,w)) = q(1-q)\sum_{R\in\{0,1\}^\ell} w^r P_R(q,w) \quad (10)$$

where

$$P_R(q,w) = \sum_{i=1}^{p}(w+1)^{c_i}\left((w+1)^{a_i} - 1\right)\left((w+1)^{b_i} - 1\right)\prod_{j<i} S_{m'_j,n'_j}(q,w)\prod_{j>i} S_{m_j,n_j}(q,w) \quad (11)$$

with $a_i, b_i, c_i, S_{m_j,n_j}(q,w), S_{m'_j,n'_j}(q,w)$ as previously defined.

4 Compression and the Tutte polynomial

The q-state Potts model $Z_G(q,w)$ is equivalent to the Tutte polynomial $T_G(x,y)$ by the change of variables

$$T_G(x,y) = (x-1)^{-\kappa(E(G))}(y-1)^{-|V(G)|} Z_G((x-1)(y-1), y-1) \quad (12)$$

where as before the notation $\kappa(G)$ indicates the number of components of graph G [41]. (As we will see this $(x-1)^{-\kappa(G)}(y-1)^{-|V(G)|}$ prefactor is the source of the $x-1$ factor appearing in the $dist_G(u,v) \geq 3$ case in Theorem 1.2.) However a simple application of (12) to Theorem 3.1 does not immediately produce the desired result, as it is not clear that the polynomials $P_R(q,w)$ remain positive polynomials under the substitutions $q = (y-1)(x-1)$ and $w = y-1$. Indeed there are two important circumstances where they do not: when a loop has been contracted in the formation of G_R and H_R then the respective term $w^r(Z_{G_R}(q,w) - Z_{H_R}(q,w))$ has $y-1$ as a factor, and when a bridge has been deleted in the formation of G_R or H_R then an $x-1$ factor may appear. If multiple loop contractions and/or bridge deletions have occurred in the formation of G_R or H_R, then additional $y-1$ or $x-1$ factors appear, and this is true even after the prefactor $(x-1)^{-\kappa(E(G))}(y-1)^{-|V(G)|}$ is accounted for.

These terms with $x-1$ or $y-1$ as a factor are not difficult to account for, however. When terms appear that correspond to loop contractions or bridge deletions, it is also the case that additional terms appear, identical to these 'problematic' terms except with one less $y-1$ or $x-1$ factor appropriately. Combining terms then reduces the number of $y-1$ or $x-1$ factors appearing by one. As we will show, we may repeat this process to eventually create a new sum for which each term is a non-negative polynomial. In fact, since this eventual sum with combined terms now is constructed using only those vectors in $\mathcal{R}$ where no loops are contracted and no bridges deleted, this new sum is precisely the sum that results by "deleting-contracting every edge of S" using the traditional Tutte polynomial deletion-contraction rule, which does not permit those operations.

To help identify and track these terms we introduce the following terminology and notation, which builds on the notation used in the proof of Theorem 3.1. For fixed R and for each $i = 1, \ldots, \ell$, let G_R^i, H_R^i denote respectively the graphs G and H after the first $i-1$ operations of R (deletion of e_j if $r_j = 0$ and contraction of e_j if $r_j = 1$) have been performed. We say *edge e_i is a loop (resp. bridge) in the formation of G_R* if the edge e_i is a loop (resp. bridge) in G_R^i. Note that edges of S that are loops or bridges of G will also be loops or bridges in the formation of G_R. We make a similar definition for the formation of H_R. We say that $R, R' \in \{0,1\}^\ell$ *i-correspond* if R and R' are identical binary vectors except at coordinate r_i.

The relevant term combinations are contained in (13) and (14) below, which for reference we collect in the following lemma. Note that (13) effectively combines a loop deletion term and loop contraction term into a single loop deletion term, and (14) combines a bridge deletion and contraction term into a bridge contraction.

Lemma 4.1. *Say R_0 and R_1 i-correspond, with $r_i = 0$ in R_0 and $r_i = 1$ in R_1, with r the number of contractions taking place under R_0. If e_i is a loop in both $G_{R_1}^i$ and $H_{R_1}^i$ then*

$$w^r(Z_{G_{R_0}}(q,w) - Z_{H_{R_0}}(q,w)) + w^{r+1}(Z_{G_{R_1}}(q,w) - Z_{H_{R_1}}(q,w)) = (1+w)w^r(Z_{G_{R_0}}(q,w) - Z_{H_{R_0}}(q,w)). \tag{13}$$

If e_i is a bridge in $G_{R_0}^i$ and $H_{R_0}^i$, then

$$w^r(Z_{G_{R_0}}(q,w) - Z_{H_{R_0}}(q,w)) + w^{r+1}(Z_{G_{R_1}}(q,w) - Z_{H_{R_1}}(q,w)) = (q+w)w^r(Z_{G_{R_1}}(q,w) - Z_{H_{R_1}}(q,w)). \tag{14}$$

Proof. Since R_0, R_1 i-correspond then e_i is a loop (resp. bridge) in $G_{R_1}^i$ if and only if e_i is a loop (resp. bridge) in $G_{R_0}^i$, and a similar statement holds with H in place of G. Now for the loop equation, since R_0, R_1 i-correspond, and deleting and contracting a loop result in the same graph, we have $G_{R_1} = H_{R_1}$ and $G_{R_0} = H_{R_0}$ and the result follows. For the bridge equation, deleting a bridge results in the same block structure as contracting the bridge, but does increase the number of components by one. By (3) and (4) then, here we have $qZ_{G_{R_1}}(q,w) = Z_{G_{R_0}}(q,w)$ and $qZ_{H_{R_1}}(q,w) = Z_{H_{R_0}}(q,w)$ and the result follows. $\square$

The previous lemma does not address the question of whether it is possible that e_i could be a bridge or a loop in one of G_R^i or H_R^i but not in the other. It is in fact possible for e_i to be a bridge in G_R^i but not in H_R^i; for a simple example consider $G = P_4$, the path on four vertices, and let u, v be opposite ends of the path. However, this type of situation is the only possibility.

Lemma 4.2. *The edge e_i is a loop in G_R^i if and only if it is a loop in H_R^i. If e_i is a bridge in H_R^i, then e_i is a bridge in G_R^i as well.*

Proof. A loop, say e_i, is contracted in the formation of G_R if and only if e_i is a loop contracted in the formation of H_R as well: if e_i is a contracted loop in the formation of G_R then every edge of some cycle of $G - u - v$ was contracted, and since $G - u - v = H - u - v$ this implies that the same cycle was contracted in the formation of H_R, and vice-versa.

If e_i is a bridge in H_R^i we may assume that $H_R^i - e_i$ consists of two components, say C and C', and that $v \in V(C)$. But then necessarily $V(C') \subseteq A_{\overline{uv}}^i$, where $A_{\overline{uv}}^i$ is $A_{\overline{uv}}$ after the first $i-1$ operations of R have been performed. This implies that no edge except e_i exists between $V(C)$ and $V(C')$ in G_R^i as well, and thus e_i is a bridge there too. $\square$

And moreover, we may effectively ignore the case when e_i is a bridge in G_R^i but not in H_R^i, as our final lemma shows.

Lemma 4.3. *Say R is such that $r_i = 0$, and e_i is a bridge in G_R^i but is not a bridge in H_R^i. Then $Z_{G_R}(q,w) - Z_{H_R}(q,w) = 0$.*

Proof. Necessarily e_i has in G_R^i one endvertex in $A_{u\overline{v}}$ and the other endvertex in $A_{\overline{u}v}$, and we may assume that $G_R^\ell - e_i$ consists of two components C and C' with $v \in V(C)$ and $u \in V(C')$. This implies that $H_R^i - e_i$ consists of two components with very specific structures: one component is the isolate u, and in the other component v is a cutvertex between blocks isomorphic to C and C'. Therefore, using (3) and (4), we have

$$Z_{H_R^i - e_i}(q,w) = q\left(\frac{1}{q} Z_C(q,w) Z_{C'}(q,w)\right) = Z_C(q,w)Z_{C'}(q,w) = Z_{G_R^i - e_i}(q,w)$$

and the result follows. $\square$

We are now ready to prove our main theorem.

Theorem 1.2 Let G be a connected graph, and let $u, v \in V(G)$. If $dist_G(u,v) \leq 2$ then

$$T_G(x,y) - T_{G_{u\to v}}(x,y) = (x+y-xy)P_{u,v}(x,y)$$

and if $dist_G(u,v) \geq 3$ then

$$T_G(x,y) - (x-1)T_{G_{u\to v}}(x,y) = (x+y-xy)P_{u,v}(x,y)$$

where $P_{u,v}(x,y)$ is a polynomial with non-negative coefficients.

Proof. The cases $uv \in E(G)$ and $S = \emptyset$ are dealt with the same way as in Theorem 3.1. As in that proof, letting $H = G_{u\to v}$ we have

$$Z_G(q,w) - Z_H(q,w) = \sum_{R\in\{0,1\}^\ell} w^r (Z_{G_R}(q,w) - Z_{H_R}(q,w)) \tag{15}$$

where r is the number of contractions that took place under R in the formation of G_R, H_R. We now use the previous lemmas to prove the following claim.

Claim 4.4. *Equation (15) may be rewritten as*

$$Z_G(q,w) - Z_H(q,w) = \sum_{R\in\mathcal{R}''} w^{|V(G)|-|V(G_R)|-t}(w+1)^s(q+w)^t(Z_{G_R}(q,w) - Z_{H_R}(q,w)) \tag{16}$$

where s is the number of loops deleted and t is the number of bridges contracted in the formation of G_R (and thus H_R as well), and $\mathcal{R}''$ consists of those $R \in \{0,1\}^\ell$ such that no loops were contracted and no bridges deleted in the formation of G_R (and thus H_R as well).

Proof. In the ordering $\{e_1, \ldots, e_\ell\}$ on S let i be the first index at which at least one $R \in \mathcal{R}$ has an edge e_i such that e_i will be a contracted loop or deleted bridge in G_R^i, and say there are $k \geq 1$ such R. For any individual such R, by Lemmas 4.2 and 4.3, when e_i will be a contracted loop (resp. deleted bridge) in G_R^i we may assume that e_i is also a contracted loop (resp. deleted bridge) in H_R^i.

The i-corresponding vectors for the k distinct R are necessarily distinct, and so using Lemma 4.1 we may combine all k such R terms with the k i-corresponding R' terms as in Lemma 4.1. Since after combining terms some $R \in \mathcal{R}$ now have no terms represented in the sum, now let $\mathcal{R}_i$ denote those $R \in \mathcal{R}$ that remain after all term combinations at index i have been done.

Now let $j > i$ be the next such index at which at least one $R \in \mathcal{R}_j$ has an edge e_j such that e_j will be a contracted loop (resp. deleted bridge) in G_R^j. (It is possible that one or more of the terms represented by these R are the product of a previous term combination.) We claim that the j-corresponding vector R' for any such R still exists in $\mathcal{R}_i$. Say to the contrary that there is $R \in \mathcal{R}_i$ with $R' \notin \mathcal{R}_i$. Then R' must have been eliminated as a result of using Lemma 4.1 during the previous combination of terms at index i. But since R, R' i-correspond then $G_{R'}^i$ and G_R^i are identical, which would imply that the term represented by R would have been eliminated at index i, a contradiction. Hence for any individual $R \in \mathcal{R}_i$ we may combine terms. As before if there are $k' \geq 1$ such $R \in \mathcal{R}_i$ then the k' i-corresponding vectors for these exist and, as before, are necessarily distinct, and so all k' pairs of terms combined as well. The same argument applies at further indices, and so we may iterate this process until we reach $\mathcal{R}_\ell = \mathcal{R}''$ and no further loop contractions or bridge deletions are present.

Examining Lemma 4.1, we see that for any $R \in \mathcal{R}''$ each individual loop deletion has resulted in one $1 + w$ factor appearing, and at each bridge contraction a $q + w$ factor has replaced a w factor. Since there are $|V(G)| - |V(G_R)|$ ones in R, the equation (16) follows. □

An important point here is that, since $\mathcal{R}''$ in (16) does not permit bridge deletions and G is assumed to be connected, we now have that each G_R is connected. This further implies that when $dist_G(u, v) \leq 2$ we also have H_R connected and when $dist_G(u, v) \geq 3$ we have that H_R has two components one of which is the isolate u. In particular, we emphasize that for any $R \in \mathcal{R}''$, none of the vertices $\{x_i\}_{i=1}^p$ of the G_R, H_R graphs are isolates.

For any individual R the calculations of $Z_{G_R}(q, w) - Z_{H_R}(q, w)$ in the proof of Theorem 3.1 still hold, and from (10) and (11) and (16) above we obtain that

$$Z_G(q, w) - Z_H(q, w) = q(1 - q) \sum_{R \in \mathcal{R}''} w^{|V(G)| - |V(G_R)| - t} (w + 1)^s (q + w)^t P_R(q, w) \tag{17}$$

where

$$P_R(q, w) = \sum_{i=1}^{p} (w + 1)^{c_i} \Big((w + 1)^{a_i} - 1 \Big) \Big((w + 1)^{b_i} - 1 \Big) \prod_{j<i} S_{m'_j, n'_j}(q, w) \prod_{j>i} S_{m_j, n_j}(q, w) \tag{18}$$

with $a_i, b_i, c_i, m_i, n_i, m'_i, n'_i$ defined as in the proof of Theorem 3.1. We are now in a position to apply the change of variables (12) by multiplying both sides of (17) by $(x - 1)^{-1}(y - 1)^{-|V(G)|}$ and substituting for q and w.

Doing so on the left-hand side of the equation, when $dist_G(u, v) \leq 2$ then $\kappa(E(G)) = \kappa(E(H)) = 1$, and clearly $|V(G)| = |V(H)|$, so here the left side becomes $T_G(x, y) - T_H(x, y)$. When $dist_G(u, v) \geq 3$ then $\kappa(E(G)) = 1$ and $\kappa(E(H)) = 2$ since u has been isolated. So here writing $(x - 1)^{-1}$ as $(x - 1)^{-2}(x - 1)$ the left side becomes $T_G(x, y) - (x - 1)T_H(x, y)$.

We turn now to the right side of (17). We first consider its form after substituting for q and w, leaving the multiplication by the prefactor $(x - 1)^{-\kappa(E(G))}(y - 1)^{-|V(G)|}$ as a last step. After the substitution $w = y - 1$ the expression $(w+1)^{c_i}((w+1)^{a_i} - 1)((w+1)^{b_i} - 1)$ becomes

$$y^{c_i}\left(y^{a_i} - 1\right)\left(y^{b_i} - 1\right) = (y-1)^2 y^{c_i}\left(\sum_{k=0}^{a_i-1} y^k\right)\left(\sum_{k=0}^{b_i-1} y^k\right)$$

where it is understood that $\sum_{k=0}^{-1} y^k = 0$. After the substitutions $q = (x-1)(y-1)$ and $w = y - 1$, the $S_{m_j,n_j}(q,w)$ expression appearing in (18) becomes

$$(x-1)(y-1) + y^{m_j} + y^{n_j} - 2 = (y-1)\left(x - 1 + \sum_{k=0}^{m_j-1} y^k + \sum_{k=0}^{n_j-1} y^k\right)$$

where again $\sum_{k=0}^{-1} y^k = 0$, and a similar expression holds for $S_{m'_j,n'_j}(q,w)$. Finally, after substitution the expression $1 - q$ simplifies to $x + y - xy$, the expression $w + 1$ simplifies to y, and the expression $q + w$ simplifies to $x(y-1)$. Therefore after these substitutions for q and w are complete the right-hand side of equation (17) becomes

$$\begin{aligned}
&(x-1)(y-1)(x+y-xy)\sum_{R\in\mathcal{R}''}(y-1)^{|V(G)|-|V(G_R)|-t} y^s x^t (y-1)^t \sum_{i=1}^{p}\Bigg\{(y-1)^2 y^{c_i}\left(\sum_{k=0}^{a_i-1} y^k\right)\left(\sum_{k=0}^{b_i-1} y^k\right) \\
&\quad \prod_{j<i}\left[(y-1)\left(x - 1 + \sum_{k=0}^{m'_j-1} y^k + \sum_{k=0}^{n'_j-1} y^k\right)\right]\prod_{j>i}\left[(y-1)\left(x - 1 + \sum_{k=0}^{m_j-1} y^k + \sum_{k=0}^{n_j-1} y^k\right)\right]\Bigg\} \\
&= (x-1)(y-1)^{p+2+|V(G)|-|V(G_R)|}(x+y-xy)\sum_{R\in\mathcal{R}''} y^s x^t \sum_{i=1}^{p}\Bigg[y^{c_i}\left(\sum_{k=0}^{a_i-1} y^k\right)\left(\sum_{k=0}^{b_i-1} y^k\right) \\
&\quad \prod_{j<i}\left(x - 1 + \sum_{k=0}^{m'_j-1} y^k + \sum_{k=0}^{n'_j-1} y^k\right)\prod_{j>i}\left(x - 1 + \sum_{k=0}^{m_j-1} y^k + \sum_{k=0}^{n_j-1} y^k\right)\Bigg]
\end{aligned}$$

which, recalling that $|V(G_R)| = p + 2$, gives us

$$\begin{aligned}
&= (x-1)(y-1)^{|V(G)|}(x+y-xy)\sum_{R\in\mathcal{R}''} y^s x^t \sum_{i=1}^{p}\Bigg[y^{c_i}\left(\sum_{k=0}^{a_i-1} y^k\right)\left(\sum_{k=0}^{b_i-1} y^k\right) \\
&\quad \prod_{j<i}\left(x - 1 + \sum_{k=0}^{m'_j-1} y^k + \sum_{k=0}^{n_j-1} y^k\right)\prod_{j>i}\left(x - 1 + \sum_{k=0}^{m_j-1} y^k + \sum_{k=0}^{n_j-1} y^k\right)\Bigg]
\end{aligned}$$

After multiplying by the prefactor $(x-1)^{-1}(y-1)^{-|V(G)|}$, then, we obtain that the right-hand side of (17) is

$$(x+y-xy)\sum_{R\in\mathcal{R}''} P_R(x,y)$$

where each $P_R(x,y)$ is given by

$$P_R(x,y) = y^s x^t \sum_{i=1}^{p} \Bigg[y^{c_i} \left(\sum_{k=0}^{a_i-1} y^k \right) \left(\sum_{k=0}^{b_i-1} y^k \right) \prod_{j<i} \left(x - 1 + \sum_{k=0}^{m'_j-1} y^k + \sum_{k=0}^{n'_j-1} y^k \right) \prod_{j>i} \left(x - 1 + \sum_{k=0}^{m_j-1} y^k + \sum_{k=0}^{n_j-1} y^k \right) \Bigg]. \tag{19}$$

Looking at the bracketed expression, each polynomial $P_R(x,y)$ is non-negative if for each $i = 1, \ldots, p$ at least one of m_i, n_i and at least one of m'_i, n'_i are non-zero. But this must be true since for each $R \in \mathcal{R}''$ none of the vertices $\{x_i\}_{i=1}^p$ in the non-u,v partite set are isolates. Thus each $P_R(x,y)$ is a non-negative polynomial, which means $P_{u,v}(x,y)$ is as well, and the proof is done. □

We can also determine when the polynomial $P_{u,v}(x,y)$ in Theorem 1.2 has positive coefficients, and are also able to specify when $P_{u,v}(x,y)$ is not divisible by x or y, useful facts since the Tutte polynomial has a number of interesting evaluations when $x = 0$ and when $y = 0$.

Corollary 4.5. *Let G be a connected graph with no loops or bridges, with $u, v \in V(G)$, and let $P_{u,v}(x,y)$ be defined as in the previous theorem. Then*

(a) *$P_{u,v}(x,y)$ is not divisible by x if and only if there exists a path in $G - u - v$ between $A_{u\overline{v}}$ and $A_{\overline{u}v}$, and*

(b) *$P_{u,v}(x,y)$ is not divisible by y if and only if there exists a path in $G - u - v$ between $A_{u\overline{v}}$ and $A_{\overline{u}v}$ that does not include any vertices of A_{uv}.*

In particular, $P_{u,v}(x,y)$ has at least one positive coefficient if and only if there exists a path in $G - u - v$ between $A_{u\overline{v}}$ and $A_{\overline{u}v}$, and $P_{u,v}(x,y)$ has a positive constant term if and only if such a path exists that avoids A_{uv}.

Proof. From (19) in the previous proof we have that $P_{u,v}(x,y) = \sum_{R \in \mathcal{R}''} P_R(x,y)$, where, in terms of a_i, b_i, c_i,

$$P_R(x,y) = y^s x^t \sum_{i=1}^{p} \Bigg[y^{c_i} \left(\sum_{k=0}^{a_i-1} y^k \right) \left(\sum_{k=0}^{b_i-1} y^k \right) \prod_{j<i} \left(x - 1 + \sum_{k=0}^{c_j-1} y^k + \sum_{k=0}^{a_j+b_j+c_j-1} y^k \right) \prod_{j>i} \left(x - 1 + \sum_{k=0}^{a_j+c_j-1} y^k + \sum_{k=0}^{b_j+c_j-1} y^k \right) \Bigg].$$

For part (a), first suppose that a path between $A_{u\overline{v}}$ and $A_{\overline{u}v}$ in G exists. Let F be a maximal spanning forest of $G - u - v$ that contains the path. By reordering the edges of $G - u - v$ if necessary we may assume that the edges of F are the first edges in the edge order. Let R be the binary vector with $r_i = 1$ whenever $e_i \in E(F)$ and $r_i = 0$ when $e_i \notin E(F)$. By maximality of F and choice of the edge ordering we have $R \in \mathcal{R}''$, i.e. no loops are contracted and no bridges deleted in the formation of G_R.

Since G itself is bridgeless, the edge ordering ensures no bridges are contracted in the formation of G_R, and so we have $t = 0$ in $P_R(x,y)$. Now let x_i denote the vertex in G_R resulting from the contraction of the component of F containing the path from $A_{u\overline{v}}$ and $A_{\overline{u}v}$, we have $a_i \geq 1$ and $b_i \geq 1$. It is easy to see this implies that none of the parenthetical expressions above may be zero or x, and hence $P_R(x, y)$ is non-zero and not divisible by x. Since $P_{u,v}(x, y)$ is the sum of non-negative polynomials, this implies $P_{u,v}(x, y)$ is non-zero and not divisible by x. In particular, $P_{u,v}(x, y)$ is not the zero polynomial, and thus positive. Similarly in the other direction, if $P_{u,v}(x, y)$ is not divisible by x then some $P_R(x, y)$ must be positive and not divisible by x. This implies an R for which $a_i \geq 1$ and $b_i \geq 1$, and thus the existence of the given path.

For part (b), if the given path exists we take F to be a maximal spanning forest of $G-u-v-A_{uv}$ that contains the path. Now choose an edge ordering such that the edges of F are last in the edge order, and again let R be the binary vector with $r_i = 1$ whenever $e_i \in E(F)$ and $r_i = 0$ when $e_i \notin E(F)$. We have $R \in \mathcal{R}''$ since no loops are contracted and no bridges deleted in the formation of G_R. Furthermore, the edge ordering ensures no loops are deleted in the formation of G_R and so $s = 0$ in $P_R(x, y)$, and for the vertex x_i in G_R which corresponds to the vertices of the contracted path we have $a_i \geq 1$, $b_i \geq 1$ and $c_i = 0$. From this it is easy to see that $P_R(x, y)$, and hence $P_{u,v}(x, y)$, is not divisible by y. The converse follows in a parallel fashion as for the x variable. $\square$

5 The (n, m) Tutte polynomial poset

If G, H are two connected graphs with n vertices and m edges, then the difference of their Tutte polynomials must necessarily have $x+y-xy$ as a factor. To see this, recall that the Tutte polynomial of G may be defined as

$$T_G(x, y) = \sum_{A \subseteq E(G)} (x-1)^{\kappa(A)-\kappa(E(G))}(y-1)^{|A|-n+\kappa(A)}$$

where $\kappa(A)$ is the number of components of the graph defined by the edges of A [20]. Since $|E(G)| = |E(H)| = m$ there is a bijection taking any subset $A_G \subseteq E(G)$ to a subset $A_H \subseteq E(H)$ with $|A_G| = |A_H|$. For A_G, A_H corresponding in this bijection, the difference of the corresponding terms in the definition above is

$$\begin{aligned}(x-1)^{\kappa(A_G)-1}&(y-1)^{|A_G|-n+\kappa(A_G)} - (x-1)^{\kappa(A_H)-1}(y-1)^{|A_H|-n+\kappa(A_H)} \\ &= (x-1)^{\kappa(A_H)-1}(y-1)^{|A_H|-n+\kappa(A_H)}\left(((x-1)(y-1))^{\kappa(A_G)-\kappa(A_H)} - 1\right)\end{aligned}$$

where we have taken $\kappa(A_G) \geq \kappa(A_H)$. (If $\kappa(A_G) \leq \kappa(A_H)$ then the same expression results with the G and H subscripts reversed.) A simple induction now shows that $((x-1)(y-1))^k - 1$ has a factor of $x + y - xy$ for any $k \geq 1$, and the result follows.

The key feature of Theorem 1.2, then, is that the polynomial $P_{u,v}(x, y)$ is a positive polynomial. $T_G(x, y)$ is in a natural sense "larger" than $T_{G_{u\to v}}(x, y)$ then, and as we will see it is this relationship that carries over to a wide array of graph parameters associated with the Tutte polynomial. To capture this phenomenon we make the following definitions.

Definition 5.1. *Let $\mathcal{G}_{n,m}$ denote the set of simple connected graphs with n vertices and m edges. The (n, m)* Tutte polynomial poset *$(\mathcal{G}_{n,m}, \preccurlyeq)$ is the poset defined on $\mathcal{G}_{n,m}$ by $H \preccurlyeq G$ if and only if*

$$T_G(x, y) - T_H(x, y) = (x + y - xy)P(x, y)$$

for some polynomial $P(x, y)$ with non-negative coefficients. If $P(x, y) \neq 0$ we may indicate this by $H \prec G$.

If $P(x,y)=0$ then $T_G(x,y)=T_H(x,y)$ and G and H are *Tutte polynomial equivalent* or *T-equivalent* graphs. If $G \preccurlyeq H$ and $H \preccurlyeq G$ we may also write $G \cong H$ to indicate G, H are Tutte polynomial equivalent. The (n,m) Tutte polynomial poset thus is more precisely a poset on the equivalence classes of the graphs of $\mathcal{G}_{n,m}$ with G, H equivalent if and only if T-equivalent, but there will be no problem if we allow any representative of an equivalence class to represent the class as a whole. We note that, while the $dist_G(u,v) \leq 2$ case of Theorem 1.2 shows that compression is a tool for moving downward in the Tutte polynomial poset, the $G \preccurlyeq H$ relation is more general than compression. In the next section a small (n,m) Tutte polynomial poset, the $(6,9)$ Tutte polynomial poset, is given which illustrates this.

The (n,m) Tutte polynomial poset relation $\preccurlyeq$ appears to capture much of the graphical information carried by the Tutte polynomial, in that $H \preccurlyeq G$ typically implies $p(H) \leq p(G)$ when p is a graphical parameter that can be obtained from the Tutte polynomial of a graph. In the rest of this section we give some examples of these parameters; these examples are by no means exhaustive but hopefully justify our definition by giving some idea of the scope of the parameters in question.

We begin by considering evaluations of the Tutte polynomial.

Theorem 5.2. *Let $x, y \geq 0$, and let $G, H \in \mathcal{G}_{n,m}$. If $H \preccurlyeq G$ then*

$$T_H(x,y) \leq T_G(x,y)$$

when $(x-1)(y-1) < 1$, and

$$T_G(x,y) \leq T_H(x,y)$$

when $(x-1)(y-1) > 1$.

Proof. If $H \preccurlyeq G$ then

$$T_G(x,y) - T_H(x,y) = (x+y-xy)P(x,y) = (1-(x-1)(y-1))P(x,y)$$

where $P(x,y)$ is a polynomial with non-negative coefficients. In the first quadrant $P(x,y) \geq 0$, and the result follows. $\square$

We remark that for general graphs the first quadrant boundary of Theorem 5.2 is best possible in the following sense. If G' indicates the graph G with a pendant edge appended, then $T_{G'}(x,y) = xT_G(x,y)$, and therefore appending a pendant edge to both G and H would negate any given evaluation in the second and third quadrant (where x is negative). Thus for any point (x,y) in those quadrants the inequality $T_H(x,y) \leq T_G(x,y)$, for example, would imply the opposite inequality $T_{H'}(x,y) \geq T_{G'}(x,y)$ for the appended graphs. Similarly appending a loop would reverse any inequality in the third and fourth quadrants where y is negative.

Evaluating $T_G(x,y)$ at $(1,1)$ gives the number of spanning trees of G and evaluations on the half-line above $(1,1)$ correspond to evaluations of the all-terminal reliability polynomial $Rel_G(p)$ at various values of p. A wide variety of other graph parameters, however, also appear here. Among the most prominent, the spanning forests of G are enumerated at $(2,1)$ and spanning connected subgraphs at $(1,2)$ (see for instance [20]), and a number of evaluations related to orientations also appear here: the number of acyclic orientations can be found at $(2,0)$ [42], totally cyclic orientations at $(0,2)$ [23], acyclic orientations with a single source at $(1,0)$ [30], and score vectors of orientations at $(2,1)$ [43]. Most recently a number of enumerations related to partial orientations—where some edges of G are oriented but others left unoriented—have been shown to correspond to (a constant multiple times) evaluations at $(3, 1/2)$, $(1/2, 3)$, $(3, 1)$, $(1, 3)$ and others [1]. All of these fall in the region where $(x-1)(y-1) < 1$, so $H \preccurlyeq G$ here implies that $p(H) \leq p(G)$ for p any of these parameters. On the other hand, the positive branches of the hyperbolas $(x-1)(y-1) = q$ for $q > 1$ all fall in the region where $(x-1)(y-1) > 1$. Evaluations along these correspond to evaluations of the q-state Potts model $Z_G(q, w)$ for various $q \geq 1$ [45], and so $H \preccurlyeq G$ implies $Z_H(q, w) \geq Z_G(q, w)$ here. (This is one of the few examples we can find of a widely known graph parameter which acts in the opposite direction of the relation $\preccurlyeq$.)

Since $G_{u \to v} \preccurlyeq G$ when $dist_G(u, v) \leq 2$, the following corollary is immediate, with the conditions for strict inequality following directly from Corollary 4.5.

Corollary 5.3. *Let $x, y \geq 0$. Let G be a connected graph with $u, v \in V(G)$ be such that $dist_G(u, v) \leq 2$. Then*

$$T_{G_{u \to v}}(x, y) \leq T_G(x, y)$$

when $(x-1)(y-1) < 1$, and

$$T_G(x, y) \leq T_{G_{u \to v}}(x, y)$$

when $(x-1)(y-1) > 1$. Furthermore if G has no loops or bridges and a path from $A_{u\overline{v}}$ to $A_{\overline{u}v}$ exists then the inequalities are strict whenever $y > 0$, and if G has no loops or bridges and such a path exists avoiding A_{uv} then the inequalities are strict for all $x, y \geq 0$.

Corollary 5.3 contains within it the compression results of [5, 38, 28] on spanning trees and reliability, but encompasses quite a bit more. Compression in fact decreases all of the evaluations previously mentioned, except evaluations of the q-state Potts model, which it increases. (This last fact is probably more clearly shown by Theorem 3.1.) We also note that when $x \geq 2$ then the inequalities above also hold for compressions at $dist_G(u, v) \geq 3$. This is clear from examination of the $dist_G(x, y) \geq 3$ equation of Theorem 1.2.

We turn now to specializations of the Tutte polynomial, that is, other polynomials of graphs that may be obtained from the Tutte polynomial. Any evaluation of a polynomial specialization of $T_G(x, y)$ corresponds, of course, to some evaluation of $T_G(x, y)$, and so we focus here instead on another aspect of polynomial specializations, their coefficients. We begin with generating functions. A number of specializations of the Tutte polynomial $T_G(x, y)$ produce generating functions whose coefficients correspond to certain parameters of G. For example, it is well-known [20] that for a connected graph G,

$$tT_G(t+1, 1) = \sum_{i=1}^{n} f_i(G) t^i$$

is a generating function with each $f_i(G)$ equal to the number of forests of G with i components, and

$$T_G(1, t+1) = \sum_{i=0}^{m-n+1} g_i(G) t^i$$

is a generating function with each $g_i(G)$ equal to the number of spanning connected subgraphs of G with $i + n - 1$ edges. Using this notation, the following is a straightforward consequence.

Theorem 5.4. *If $H \preccurlyeq G$ then*

$$f_i(H) \leq f_i(G) \quad \text{and} \quad g_j(H) \leq g_j(G)$$

for all $i = 1, \ldots, n$ and $j = 0, \ldots, m-n+1$. In particular, when $dist_G(u, v) \leq 2$ then the above holds for $H = G_{u \to v}$.

Proof. Substituting either $x = t+1, y = 1$ or $x = 1, y = t+1$ into $x+y-xy$ yields 1, and so

$$T_G(t+1,1) - T_H(t+1,1) = P(t+1,1)$$

and

$$T_G(1,t+1) - T_H(1,t+1) = P(1,t+1)$$

where P has all non-negative coefficients. Therefore the coefficients of the generating functions given cannot have increased after compression, giving the result. $\square$

The conditions for strict inequality given by Corollary 4.5 are somewhat less useful here; when met, we are only guaranteed that at least one of the generating function coefficients has decreased, and which coefficient or coefficients is not immediately apparent.

Many other generating functions of interest are specializations of the Tutte polynomial and, as in the proofs of Theorems 5.2 and 5.4, as long as positive expressions are substituted for x, y these essentially reduce to checking what happens with the $x+y-xy$ factor in Theorem 1.2. To illustrate with a few more examples, the generating function of the number of critical configurations of level i of the Abelian sandpile model on G is given by $T_G(1,t)$ [33], or looking again at orientations, certain types of partial orientations have generating functions equivalent to (a constant multiple times) $T_G(2+t, 1/(1+t))$ and $T_G(1+2t, 1/2)$ [21]. It is easy to check that the resulting $x+y-xy$ factor for these substitutions reduces to a positive function, and hence $H \preccurlyeq G$ implies a decrease in these parameters as well. A more complicated example involves fourientations of G, where each edge may be oriented in either direction, in both directions, or left unoriented. In [2] the trivariate generating functions

$$(j+k)^{n-1}(k+l)^{m-n+1}T_G\left(\frac{\alpha k + \beta l + j}{j+k}, \frac{\gamma k + l + \delta j}{k+l}\right)$$

for $\alpha, \gamma \in \{0,1,2\}$ and $\beta, \delta \in \{0,1\}$ were shown to enumerate a wide variety of different types of fourientations, generalizing a number of previous results on orientations and partial orientations of G. The expressions for x, y above are clearly positive, and after substituting those expressions in $x+y-xy$ and some algebraic simplification (which we omit) we obtain

$$\frac{(\gamma + \alpha - \alpha\gamma)k^2 + (1 + \beta - \beta\gamma)kl + (1 + \delta - \delta\alpha)jk + (1 - \beta\delta)jl}{(j+k)(k+l)}$$

which is non-negative for the values of $\alpha, \beta, \gamma, \delta$ permitted. Here too, then, all of the quantities enumerated by these generating functions can only decrease as we move downward in the (n,m) Tutte polynomial poset, and hence compression (at $dist_G(u,v) \leq 2$) cannot increase these quantities.

We turn now to other well-known polynomials which may be obtained as specializations of the Tutte polynomial. The all-terminal reliability polynomial, mentioned in the Introduction, is a polynomial in the variable p. In addition to the usual polynomial basis p^i, however, $Rel_G(p)$ is commonly written with respect to different bases, as the coefficients in these forms then acquire useful combinatorial information (see for example [12]). In its so-called S-form, for example, it is written as

$$Rel_G(p) = \sum_{i=0}^{m-n+1} g_i(G)p^{i+n-1}(1-p)^{m-n+1-i}$$

where the coefficients are the $g_i(G)$ defined as in Theorem 5.4, and in its H-form it is written as

$$Rel_G(p) = p^{n-1} \sum_{i=0}^{m-n+1} h_i(G)(1-p)^i. \tag{20}$$

Given these definitions it is now an immediate consequence of Theorem 5.4 that the coefficients of $Rel_G(p)$ in its S-form decrease under compression or, more generally, decrease when descending the (n,m) Tutte polynomial poset. And a result of [9], on compression's effect on the H-form coefficients of the all-terminal reliability polynomial, also becomes now a straightforward calculation.

Theorem 5.5. *Let $G, H \in \mathcal{G}_{n,m}$. If $H \preccurlyeq G$ then $h_i(H) \leq h_i(G)$ for all $i = 0, \ldots, m-n+1$, where the h_i are the coefficients of the all-terminal reliability polynomial in its H form. In particular, when $dist_G(u,v) \leq 2$ then the above holds for $H = G_{u\to v}$.*

Proof. As mentioned in the Introduction the specialization of the Tutte polynomial to the all-terminal reliability polynomial is

$$Rel_G(p) = p^{n-1}(1-p)^{m-n+1} T_G(1, 1/(1-p)).$$

Combining this and (20) we have

$$\sum_{i=0}^{m-n+1} (h_i(G) - h_i(H))(1-p)^{i-m+n-1} = T_G(1,1/(1-p)) - T_H(1,1/(1-p)) = P(1,1/(1-p))$$

where the polynomial P is as given by Theorem 1.2. Thus $H \preccurlyeq G$ implies $h_i(H) \leq h_i(G)$, and so the coefficients of the H-form of the all-terminal reliability polynomial decrease as well. $\square$

Perhaps the best-known specializations of the Tutte polynomial are the chromatic and flow polynomials. Unlike the previously seen generating functions or the S and H all-terminal reliability forms, these polynomials have negative coefficients. (It is well-known that they alternate in sign.) Here we can show that the magnitude of the coefficients decreases as we move downward in the poset. For the chromatic polynomial we can also show that this holds for compression at any u, v distance, a result first shown by Csikvári [14]. Theorem 1.2 provides a short algebraic proof of this.

Theorem 5.6. *Let $\chi_G(\lambda)$ denote the chromatic polynomial of a connected graph G and let $c_i(G)$ denote its coefficients. If $H \preccurlyeq G$ then*

$$|c_i(G)| \geq |c_i(H)|$$

for all $i = 1, \ldots, n$. In addition, for $u, v \in V(G)$ at any distance we have

$$|c_i(G)| \geq |c_i(G_{u\to v})|$$

for all $i = 1, \ldots, n$.

Proof. We only do the compression argument, as the argument for $H \preccurlyeq G$ parallels the $dist_G(u,v) \leq 2$ case. As given in the Introduction, the chromatic polynomial is

$$\chi_G(\lambda) = (-1)^{|V|-\kappa(E(G))} \lambda^{\kappa(E(G))} T_G(1-\lambda, 0).$$

Since we are only concerned with the magnitude of the coefficients, it is convenient to consider

$$\overline{\chi}_G(\lambda) = \lambda^{\kappa(E)} T_G(1+\lambda, 0)$$

which has the same coefficients as $\chi_G(\lambda)$ but with coefficients all non-negative. When $dist_G(u,v) \leq 2$, we have $\kappa(E(G)) = \kappa(E(G_{u\to v})) = 1$. By Theorem 1.2 then we have

$$\overline{\chi}_G(\lambda) - \overline{\chi}_{G_{u\to v}}(\lambda) = \lambda(T_G(1+\lambda,0) - T_{G_{u\to v}}(1+\lambda,0)) = \lambda(1+\lambda)P(1+\lambda,0)$$

which is a polynomial with non-negative coefficients as required.

When $dist_G(u,v) \geq 3$ compression from u to v renders u an isolate and we have $\kappa(E(G_{u\to u})) = 2$. By Theorem 1.2 we have

$$T_G(x,y) - (x-1)T_{G_{u\to v}}(x,y) = (x+y-xy)P(x,y).$$

Multiplying the above by λ and substituting $x = 1+\lambda, y = 0$ we obtain

$$\lambda T_G(1+\lambda,0) - \lambda^2 T_{G_{u\to v}}(1+\lambda,0) = \lambda(1+\lambda)P(1+\lambda,0)$$

or

$$\overline{\chi}_G(\lambda) - \overline{\chi}_{G_{u\to v}}(\lambda) = \lambda(1+\lambda)P(1+\lambda,0)$$

as in the previous case. □

When $dist_G(u,v) \leq 2$ the same type of result holds for the flow polynomial. (When $dist_G(u,v) \geq 3$ compression can increase the flow polynomial's coefficients. A simple example of this takes $G = P_4$, the path on four vertices, with u, v the endvertices of the path. The flow polynomial of a path is the zero polynomial, but the flow polynomial of $G_{u\to v} = C_3 \cup K_1$ is not.) In terms of the Tutte polynomial, the flow polynomial of a connected graph G is given by

$$F_G(y) = (-1)^{|E(G)|-|V(G)|+1} T_G(0, 1-y).$$

In the same manner as in the $dist_G(u,v) \leq 2$ case of the chromatic polynomial therefore we have the following, whose proof is omitted.

Theorem 5.7. *Let $F_G(y)$ denote the flow polynomial of a connected graph G, let $b_i(G)$ for $i = 1, \ldots, m-n+1$ denote its coefficients. If $H \preccurlyeq G$ then*

$$|b_i(G)| \geq |b_i(H)|$$

for all $i = 1, \ldots, |E(G)| - |V(G)| + 1$. In particular, when $dist_G(u,v) \leq 2$ then the above holds for $H = G_{u\to v}$.

The coefficients of the Tutte polynomial itself have various interpretations. Let t_{ij} denote the coefficient of the $x^i y^j$ term of $T_G(x,y)$. Then t_{ij} counts the number of spanning trees of G with certain measures of internal and external activity; we omit the definitions here, but for more details see for instance [8]. When $H \preccurlyeq G$ then by looking at the expanded form of $(x+y-xy)P(x,y)$ Theorem 1.2 gives immediately that $t_{i0}(G) \geq t_{i0}(H)$ and $t_{0j}(G) \geq t_{0j}(H)$ for any i or j in the appropriate ranges. One of these coefficients is of particular note. The coefficient of the x term and the coefficient of the y term of $T_G(x,y)$ (it is known the two coefficients must be equal, unless G has just one edge) is equal to Crapo's beta invariant $\beta(G)$ [20], first introduced in [13] in the context of matroids. When $H \preccurlyeq G$ we see that $\beta(G) - \beta(H) = c$ where c is the constant term of $P(x,y)$, which immediately gives the following. The path conditions for strict inequality again follow from Corollary 4.5.

Theorem 5.8. *If $H \preccurlyeq G$ then $\beta(H) \leq \beta(G)$. In particular, when $dist_G(u,v) \leq 2$ then $\beta(G_{u\to v}) \leq \beta(G)$, with $\beta(G_{u\to v}) < \beta(G)$ if and only if there exists a path in $G-u-v$ from $A_{u\overline{v}}$ to $A_{\overline{u}v}$ that avoids A_{uv}.*

The parameter $\beta(G)$ is also known to equal a particular evaluation of the derivative of the chromatic polynomial [20], and with regard to orientations the number $2\beta(G)$ is known to give the number of orientations of G that have a unique source and sink (over all possible sources and sinks) [31], and so $H \preccurlyeq G$ implies a decrease in those quantities.

6 Threshold graphs and the Tutte polynomial

The $dist_G(u,v) \leq 2$ case of Theorem 1.2 demonstrates that compression is a tool for descending (n,m) Tutte polynomial posets, i.e., $G_{u\to v} \preccurlyeq G$, with Corollary 4.5 giving conditions which ensure $G_{u\to v} \prec G$. To illustrate, a Hasse diagram of the $(6,9)$ Tutte polynomial poset is given in Figure 2. In Figure 2, when a Hasse diagram descent is an immediate result of a compression we have indicated those edges in red, but typically compression involves a descent of multiple levels. To mention just one example of compression effects not pictured, if we let G_{min} indicate the minimum graph in the poset, then more than half the graphs in the poset have vertices u,v such that $G_{u\to v} \cong G_{min}$.

A notable feature of Figure 2 is that the minimum graph in the $(6,9)$ Tutte polynomial poset shown is a threshold graph. Threshold graphs are a well-known and much studied class of graphs (see e.g. [32]) and there are many equivalent ways to define them. With regard to compression one of the more illuminating definitions involves a dominance relation on vertices. We say *vertex v dominates vertex u in G* if $N_G(u) \subseteq N_G[v]$, where $N_G[v]$ is the closed neighborhood $N_G[v] = N_G(v) \cup v$. A *threshold graph* can be defined as a graph G in which, given any pair of vertices $u,v \in V(G)$, either u dominates v or v dominates u. It is easy to see that when $G_{u\to v} \neq G$ then the compression of G from u to v takes two vertices $u,v \in V(G)$ which do not dominate each other and produces a new graph $G_{u\to v}$ in which v does dominate u. After compression, then, the new graph $G_{u\to v}$ will be "more threshold" than the original graph G and continued application of compression (with different pairs of vertices) can only increase this. As noted by a number of authors [5, 6, 14, 16, 17, 38], eventually this process produces a threshold graph, and if G was connected then by taking all compressions to be at distance 2 or less we can take the eventual threshold graph to be connected as well, facts which appeared as Theorem 1.1 in the Introduction.

An immediate consequence of these theorems and Theorem 1.1 then is that threshold graphs are minimal in the (n,m) Tutte polynomial posets.

Theorem 6.1. *For any n,m and any graph $G \in \mathcal{G}_{n,m}$, there exists a threshold graph $H \in \mathcal{G}_{n,m}$ such that $H \preccurlyeq G$.*

Therefore threshold graphs are extremal graphs for all the parameters considered in the previous section. To be explicit, in terms of evaluations of the Tutte polynomial we have the following.

Corollary 6.2. *Let $x,y \geq 0$. Then for any connected simple graph G there exists a connected threshold graph H with the same number of vertices and edges such that*

$$T_G(x,y) \geq T_H(x,y)$$

when $(x-1)(y-1) < 1$, and

$$T_G(x,y) \leq T_H(x,y)$$

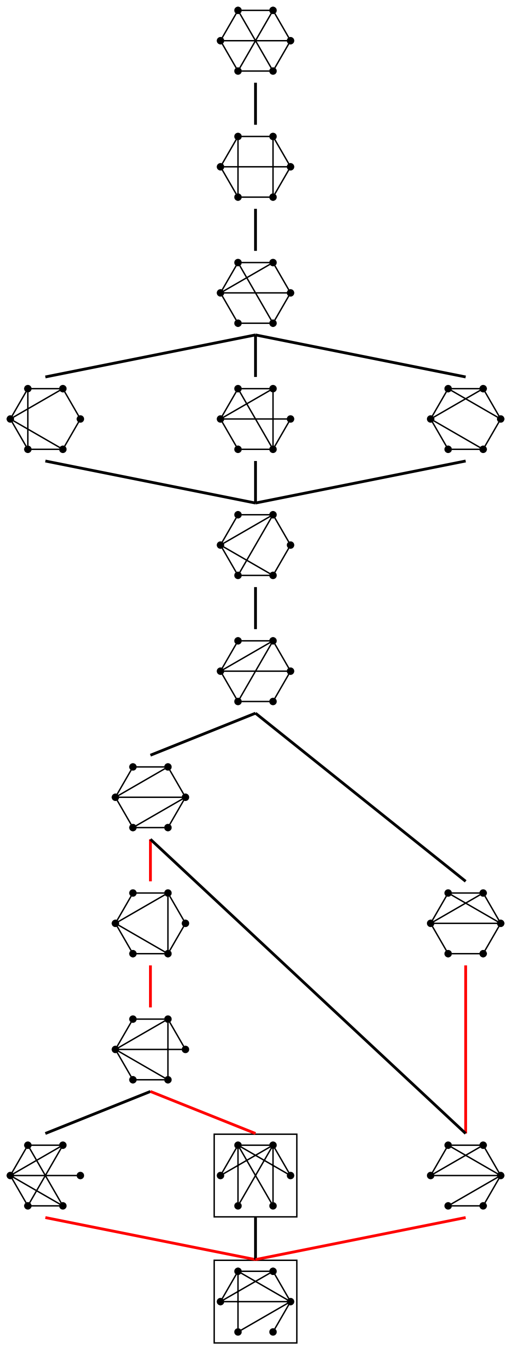

Figure 2: The $(6, 9)$ Tutte polynomial poset. If multiple graphs have the same Tutte polynomial only one graph is shown. As one moves downward in the poset, a wide variety of graph parameters associated with the Tutte polynomial decrease. The two graphs shown in boxes are threshold graphs. Red poset edges indicate that the lower graph (or a graph Tutte-equivalent to it) may be obtained from the upper graph via compression.

when $(x-1)(y-1) > 1$.

As in Theorem 5.2, this implies that threshold graphs are minimal for a number of graph parameters, i.e., for every $G \in \mathcal{G}_{n,m}$ there is a threshold $H \in \mathcal{G}_{n,m}$ such that H has fewer spanning trees, spanning forests, or spanning connected subgraphs of G; lower all-terminal reliability of G, for any edge probability p; and fewer of any of a variety of different types of orientations, partial orientations, and fourientations. The spanning tree and all-terminal reliability results implicit in Theorem 6.2 recover the extremality results of [5, 38]. On the other hand, threshold graphs are maximal for evaluations of the q-state Potts model for $q \geq 1$. The situation is similar for the specializations of the Tutte polynomial. For any connected simple graph $G \in \mathcal{G}_{n,m}$ there exists a connected threshold graph $H \in \mathcal{G}_{n,m}$ with $c_i(H) \leq c_i(G)$, where $c_i(G)$ is the magnitude of the i^{th} coefficient of any of the following polynomials: the chromatic polynomial $\chi(G)$; the flow polynomial $F(G)$; the all-terminal reliability polynomial $Rel_G(p)$ (in either its S or H forms); and all of the generating functions mentioned in the previous section. The chromatic coefficient result here recovers the extremality results of Csikvári [14] and Rodriguez and Satyanarayana [37] mentioned in the Introduction.

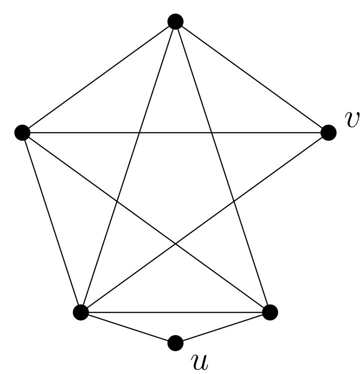

Figure 3: A planar graph G with non-planar compression $G_{u\to v}$.

In our results so far we have been careful to consider only parameters of general graphs. There are a number of other parameters, however, that may be obtained from the Tutte polynomial that are specific only to particular graph subclasses. As an example, when G is planar then evaluations of $T_G(x, y)$ along the line $x = y$ enumerate various structures associated with the medial graph of G [18, 19]. When both G and H are planar then analysis of these parameters goes along in a similar manner to what has been shown. However in the context of compression we do note that compression of a graph G from u to v may or not preserve planarity (or other graph properties) depending upon which u and v are selected. Figure 6 gives a simple example of a planar graph G with two vertices u, v that has a non-planar compression $G_{u\to v}$. It is clear how to adapt that example to obtain, for any given forbidden subgraph F, a graph G that does not contain F but whose compression $G_{u\to v}$ does contain F.

It may also be worth mentioning here, however, that for some classes of graphs it has been shown possible via judicious choice of the u, v vertices compressed upon to maintain membership in the class. For instance using this approach Bogdanowicz [7] showed that any simple 2-connected chordal graph may be transformed into a 2-connected threshold graph via a series of compressions.

Theorem 1.2 and Theorem 1.1 thus gives that threshold graphs are minimizers in this subclass of graphs as well.

Theorem 6.3. *For any 2-connected chordal graph $G \in \mathcal{G}_{n,m}$, there exists a 2-connected threshold graph $H \in \mathcal{G}_{n,m}$ such that $H \preccurlyeq G$.*

Returning now to planar graphs, one class of widely studied planar graphs are the 2-trees. A *2-tree* G_t is a graph constructed recursively by taking $G_0 = K_3$, and at each step $1 \leq i \leq t$ selecting one edge of G_{i-1} and adding a path of 2 edges between the endvertices of that edge. 2-trees are both chordal and 2-connected, and it is easily verified that there is a unique threshold 2-tree: the 2-tree in which all t of the paths have been added to the same edge of the initial triangle. Calling this 2-tree B_t, we have identified the minimizing graph for one planar class of graphs, the 2-trees.

Corollary 6.4. *If G_t is a 2-tree, then $B_t \preccurlyeq G_t$.*

In [44] it was shown that B_t was the minimum 2-tree for spanning trees. Corollary 6.4 significantly generalizes this result, and in a much easier manner.

We also briefly raise the question of maximal graphs in the (n, m) Tutte polynomial posets. This problem appears to be more difficult than the minimality problem. Some guidance here comes from work identifying extremal graphs for spanning trees and reliability. With regard to reliability, it is known for example that certain (n, m) classes have no uniformly most reliable graphs. The smallest such class, determined in [34], is $(6, 11)$, and it is not hard to verify that the $(6, 11)$ Tutte polynomial poset does not have a unique maximum element, instead having two maximal graphs. (The two graphs have $2P_3$ and $P_4 \cup P_2$ as their complements, respectively.) On the other hand, in (n, m) classes that do have uniformly most reliable graphs these graphs may be good candidates for maximum elements. These graphs necessarily also have the maximum number of spanning trees in the class, and these graphs, often called *t-optimal graphs*, may also be good candidates for maximum elements of the corresponding (n, m) Tutte polynomial posets. Less promisingly, however, t-optimal graphs are currently known only for edge dense graphs [22, 27, 36, 39], edge sparse graphs [3], or regular or almost regular graphs [11, 36], and it is not entirely clear what these maximum graphs have in common. We are not aware even of any conjectures about general features of t-optimal graphs for all (simple) values of n, m.

For the most edge dense graphs we can generalize the t-optimal results to an (n, m) Tutte polynomial poset result, which we do below, but we make no attempt to obtain any other maximal results here.

Theorem 6.5. *Let k, m, n be such that $0 \leq k \leq n/2$ and $m = \binom{n}{2} - k$. If $H \in \mathcal{G}_{n,m}$ then $H \preccurlyeq M_k$, where M_k is the complete graph K_n with any set of k independent edges removed.*

Proof. Assume to the contrary that there exists some graph in the poset that is either incomparable to M_k or greater than M_k in the poset. Let H be such a graph that is maximal in the poset. For these values of k the graph H must contain a vertex v of degree $n - 1$ and a vertex u of degree $n - 3$ or less. Since u and v are adjacent, given the disparity in degrees there must be at least two vertices, call them x, y, adjacent to v but not u. But then letting G be identical to H but with x adjacent to u instead of v, we have $G_{u \to v} = H$. Clearly there exists a path in G from $A_{u\overline{v}}$ to $A_{\overline{u}v}$, and so $H \prec G$ and H cannot be maximal in the (n, m) Tutte polynomial poset. This contradiction proves the result. $\square$

While not every (n, m) Tutte polynomial poset has a unique maximum element, we conclude however with a conjecture that for any $\mathcal{G}_{n,m}$ the (n, m) Tutte polynomial poset does have a minimum element. The graphs corresponding to these minimums must, of course, be T-equivalent to a threshold graph, and we now describe the form of the threshold graph that we believe corresponds to this minimum.

Another useful way to define threshold graphs is as a subclass of *split graphs*, graphs whose vertex sets can be partitioned into two sets, one of which induces a clique and the other of which induces an independent set. A threshold graph is a split graph in which the vertices of the independent set have nested neighborhoods [32]. Informally, our conjectured minimum graphs are obtained by making the clique part as large as possible and then, when connecting the vertices of the independent set to the clique, making as many degree one vertices as possible. Formally, let k be the least integer such that $m \geq \binom{n-k}{2} + k$. Then the graph $L_{n,m}$ is the threshold graph consisting of an $(n-k)$-clique, and $k-1$ pendant vertices and one vertex of degree $m - \binom{n-k}{2} - k + 1$ attached to the clique.

Conjecture 6.6. *For any $G \in \mathcal{G}_{n,m}$, we have $L_{n,m} \preccurlyeq G$.*

Some evidence in support of the above conjecture comes again from spanning trees, all-terminal reliability, and chromatic coefficients. In [6] it was proven that the $L_{n,m}$ graphs minimize $\tau(G)$, and in [37] the magnitudes of the coefficients of $\chi(G)$ were also shown to be minimized by the $L_{n,m}$ graphs. It is a long-standing conjecture of Boesch et al. [4] that the graphs $L_{n,m}$ minimize $Rel_G(p)$. That conjecture is still open; proving Conjecture 6.6 would of course prove that conjecture, as well as recover the spanning tree and chromatic polynomial results.

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