ON A CONJECTURE OF LEVIT AND MANDRESCU

NATHAN KAHL, HANNAH QUENSE, AND TARA WAGER

ABSTRACT. The *independence polynomial* $I(G;x)$ of a graph G is defined to be $I(G;x) = \sum_{k=1}^{\alpha(G)} s_k x^k$, where $\alpha(G)$ is the independence number of G and s_k is the number of independent sets in G of size k. The *decycling number* of a graph G, denoted $\varphi(G)$, is defined to be the minimum size of a set $S \in V(G)$ such that $G - S$ is acyclic. Levit and Mandrescu conjectured that for every positive integer k, there is a graph G with $\varphi(G) = k$ and $I(G;-1) = q$ with $|q| \leq 2^k$, and provided constructions in support of this conjecture for all $k \leq 3$, and for $k = 4$ and $q \neq \pm 13$. In this paper, using a new graph operation that we call the lateral join, we provide constructions which support the conjecture for all $k \leq 5$.

1. INTRODUCTION

For any undefined terms or notation we refer the reader to [6]. Since we will be dealing extensively with the union and join operations on graphs, we do mention that for two graphs G and H with disjoint vertex sets, we will use $G \cup H$ to denote their disjoint union and $G + H$ to denote their join. We will also use the standard abbreviations for multiple unions of the same graph, e.g., $C_4 \cup C_4 \cup C_4 = 3C_4$. A further reference on independence polynomials is the survey [3].

A set of vertices S of a graph G is *independent* if the vertices of S are pairwise nonadjacent, in which case the set S is called an *independent set*. The *independence number of* G, denoted $\alpha(G)$, is the cardinality of a maximum independence set. The *independence polynomial of* G is

$$I(G;x) = \sum_{k=0}^{\alpha(G)} s_k x^k = s_0 + s_1 x + s_2 x^2 + ... + s_{\alpha(G)} x^{\alpha(G)}$$

where s_k is the number of independent sets of order k in G. A *decycling set* of G is a set of vertices which, when removed from G, leave a graph containing no cycles, and the *decycling number* $\varphi(G)$ *of a graph* G is the order of a minimum size decycling set.

In [1], Engström proved the following bound on the independence polynomial $I(G;x)$ when evaluated at -1.

Theorem 1.1. *For any graph* G,

$$|I(G;-1)| \leq 2^{\varphi(G)}$$

where $\varphi(G)$ *is the decycling number of* G.

In [4], Levit and Mandrescu produced an elementary proof of Engström's result and gave the following conjecture.

Conjecture 1.2. For every positive integer k and each integer q such that $|q| \leq 2^k$, there is a graph G with $\varphi(G) = k$ and $I(G;-1) = q$.

For reference, if a graph G has decycling number $\varphi(G) = k$ and independence polynomial $I(G;-1) = q$, we will say that G is at *level* k and has *value* q. Let S be a set of graphs such that, if $|q| \leq 2^k$, then there exists some $G \in S$ such that $I(G;-1) = q$. When this occurs we will denote such a set as $\mathcal{G}_k$ and say $\mathcal{G}_k$ *completes* level k. In a later paper [5] (on unicyclic and well-covered graphs) Levit and Mandrescu noted that the various theorems there, together with some previous results, could complete all levels $k \leq 3$, and could also produce nearly all of $\mathcal{G}_4$, noting however that on level 4, "The case of $q = \pm 13$ has not been settled yet."

While the constructions noted by Levit and Mandrescu were various and ad hoc, in this paper, using a new graph operation we call a *lateral join*, together with the familiar operations of graph unions and graph joins, we are able to systematically settle level 4, including the the $q = \pm 13$ case, and also produce $\mathcal{G}_5$, with the exception of $q = \pm 19$. A very similar, but ad hoc, construction produces a $q = \pm 19$ graph on level 5, completing the level.

2. Preliminaries and Level 4

Our constructions are inductive, using the graphs of a given $\mathcal{G}_k$ to construct the required graphs of $\mathcal{G}_{k+1}$. To illustrate the techniques we show how to construct all of $\mathcal{G}_4$, including graphs with $q = \pm 13$, from the graphs of $\mathcal{G}_3$. We will then use the same techniques towards completing level 5.

We first gather together some useful facts on independence polynomials. It is well-known (and also easy to calculate) that

$$I(K_2;-1) = -1 \qquad \text{and} \qquad I(C_3;-1) = -2.$$

The following facts on graph unions and joins are also well-known (see for instance the survey [3]).

Theorem 2.1. *Let* G *and* H *be two graphs with disjoint vertex sets. Then*

(a) $I(G \cup H;x) = I(G;x)I(H;x)$, *and*

(b) $I(G + H;x) = I(G;x) + I(H;x) - 1$.

Regarding graph unions and joins, we note the following facts on decycling numbers.

Theorem 2.2. *Let G and H be two graphs with disjoint vertex sets. Then*

(a) $\varphi(G \cup H) = \varphi(G) + \varphi(H)$

(b) $\varphi(G + K_1) = \varphi(G)$ *if and only if* $E(G) = \emptyset$. *In particular, if G is not an edgeless graph, then* $\varphi(G + K_1) = \varphi(G) + 1$.

Proof. Part (a) is clear. For part (b), if $E(G) = \emptyset$ then $G + K_1$ is a star, and thus $\varphi(G) = \varphi(G + K_1) = 0$. Thus the "if" part of the statement is true. For the "only if" part, let $\varphi(G + K_1) = \varphi(G) = s$. If $s = 0$, then G is a forest. However, if G is a forest with any edges then $G + K_1$ has a triangle, in which case $\varphi(G + K_1) > 0 = \varphi(G)$. Thus if $s = 0$ then G must be edgeless.

We now show that when $s > 0$ then $\varphi(G + K_1) = \varphi(G) = s$ results in a contradiction. Let S be a decycling set of $G + K_1$, so that $|S| = s$, and let v denote the K_1 vertex. If $v \in S$, then $S - v$ must be a decycling set of G. Since $|S - \{v\}| = s - 1 < \varphi(G)$, this contradicts that $\varphi(G) = s$. Thus $v \notin S$. But then $E(G - S)$ must be empty, else $(G - S) + K_1$ would have a triangle, contradicting that S is a decycling set of $G + K_1$. Since $s > 0$, there must be a vertex $w \in S$. But then since $G - S$ has no edges, then $G - (S - \{w\})$ is a forest. In particular, $G - (S - \{w\})$ has no cycles, and thus $S - \{w\}$ is a decycling set of G of cardinality $s - 1$, contradicting that $\varphi(G) = s$, and ending the proof. □

Taken altogether, the previously mentioned facts give us the following useful lemma regarding independence polynomials evaluated at -1.

Lemma 2.3. *Let G be a level k graph with value q, i.e.* $\varphi(G) = k$ *and* $I(G; -1) = q$. *Then*

(a) $G \cup P_2$ *is a level k graph with value $-q$,*

(b) $G \cup C_3$ *is a level $k + 1$ graph with value $-2q$, and*

(c) If $E(G) \neq \emptyset$, *then $G + K_1$ is a level $k + 1$ graph with value $q - 1$.*

Proof. For parts (a) and (b), Lemma 2.2(a) gives

$$\varphi(G \cup P_2) = \varphi(G) + \varphi(P_2) = \varphi(G) + 0 = k$$

and

$$\varphi(G \cup C_3) = \varphi(G) + \varphi(C_3) = \varphi(G) + 1 = k + 1.$$

By Theorem 2.1(a), then

$$I(G \cup P_2; -1) = I(G; -1)I(P_2; -1) = I(G; -1)(-1) = -q$$

and

$$I(G \cup C_3; -1) = I(G; -1)I(C_3; -1) = I(G; -1)(-2) = -2q.$$

Thus $G \cup P_2$ is a level k graph with value $-q$, and $G \cup C_3$ is a level $k + 1$ graph with value $-2q$. For part (c), when G has edges Lemma 2.2(b) gives $\varphi(G + K_1) = \varphi(G) + 1$. And by Theorem 2.1b,

$$I(G + K_1; -1) = I(G; -1) + I(K_1; -1) - 1 = I(G; -1) + 0 - 1 = q - 1$$

Thus $G + K_1$ is a level $k+1$ graph with value $q-1$. □

Informally, part (a) tells us that we can "multiply q by -1" and remain on the same level; parts (b) and (c) tell us, respectively, that "multiplying q by -2" or "subtracting 1 from q" can be done, at the cost of moving up a level.

Now we apply Lemma 2.3 toward completing level 4. This requires constructing $2(2^4)+1 = 33$ graphs, for all q-values from -16 to 16. However, since the $k = 3$ case has been completed in previous work, when constructing $\mathcal{G}_4$ we may "multiply $\mathcal{G}_3$ by two" by part (b) of the lemma and consider the graphs for any $q \in \{-16, -14, -12, \ldots, 12, 14, 16\}$ to be done. By part (c) we may also "subtract one from $\mathcal{G}_3$" and consider any $q \in \{-9, -8, \ldots, 6, 7\}$ as done. And since $q = -9$ is done, by part (a) of the lemma we may also take $q = 9$ as done. To complete $\mathcal{G}_4$ then, what remains are only the graphs for the odd $q \in \{-15, -13, -11, 11, 13, 15\}$ to be constructed. And by part (a), it suffices to construct graphs for $q \in \{15, 13, 11\}$.

To do this we require one final tool, a decomposition result first due to Gutman and Harary [2].

Theorem 2.4. *Let G be a graph and v be a vertex in G. Then*

$$I(G;x) = I(G-v;x) + xI(G-N[v];x)$$

where $N[v]$ is the closed neighborhood of v in G.

Specialized to our problem, we have the following useful equation:

$$I(G;-1) = I(G-v;-1) - I(G-N[v];-1). \tag{1}$$

Our motivation for using equation (1) is the observed fact that, after applying Lemma 2.3 to $\mathcal{G}_k$, we are left only considering odd numbers. Our goal, therefore, will be to construct graphs $G \in \mathcal{G}_{k+1}$ such that

$$I(G-v;-1) = \pm 2I(G';-1)$$

for an appropriate $G' \in \mathcal{G}_k$, and

$$I(G-N[v];-1) = \pm 1,$$

thus guaranteeing that $I(G;-1)$ is odd. If $G = G' \cup C_3$ then the first condition is met, and the second condition, while not guaranteed, is likely if $G - N[v]$ is a forest. These considerations give rise to our final graph operation, the lateral join.

Definition 2.5. Let G be a graph and $S \subset V(G)$ a decycling set of G. The *first S-lateral join* of G, denoted $G_{S,l}$, consists of $G \cup C_3$ together with a new vertex v and edges joining v to each vertex of S and to any vertex of the C_3. The *second S-lateral join* of G, denoted $G_{S,2l}$, consists of $G_{S,l} \cup C_3$ together with a new vertex w and edges joining w to v, w to S, and w to any vertex of the C_3.

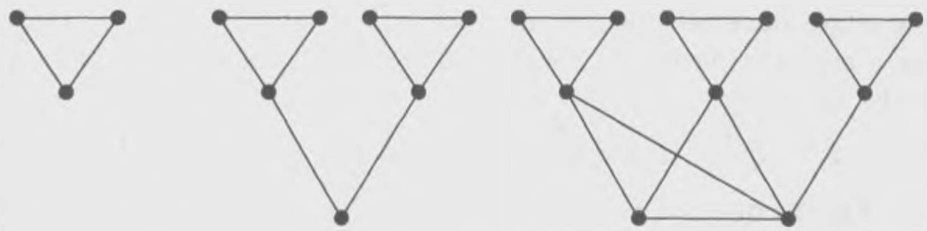

FIGURE 1. The graph C_3 with its first and second lateral joins.

We will typically shorten "first S-lateral join" to S-lateral join or simply lateral join when S is clear. The second lateral join operation is not needed to complete this level but will be used to complete level 5. We note the following relevant facts about the lateral join operations.

Lemma 2.6. *Let G be a graph, $G_{S,l}$ its first lateral join and $G_{S,2l}$ its second lateral join. Then*

(a) $\varphi(G_{S,l}) = \varphi(G) + 1$,
(b) $\varphi(G_{S,2l}) = \varphi(G_{S,l}) + 1$,
(c) $I(G_{S,l}; -1) = -2I(G; -1) + I(G - S; -1)$,
(d) $I(G_{S,2l}; -1) = -2I(G_{S,l}; -1) - I(G - S; -1)$.

Note that $G - S$ is the forest of G left when the decycling set S is deleted.

Proof. Parts (a) and (b) are clear. As in the definition, let v indicate the vertex added during the lateral join operation and w the vertex added during the second lateral join operation. Since $G_{S,l} - v = G \cup C_3$ and $G_{S,l} - N[v] = (G - S) \cup K_2$, applying equation (1) gives

$$\begin{aligned} I(G_{S,l}; -1) &= I(G \cup C_3; -1) - I((G - S) \cup K - 2; -1) \\ &= -2I(G; -1) + I(G - S; -1). \end{aligned}$$

Similarly, since $G_{S,2l} - w = G_{S,l} \cup C_3$ and $G_{S,2l} - N[w] = (G - S) \cup 2K_2$, applying equation (1) to vertex w we obtain

$$\begin{aligned} I(G_{S,2l}; -1) &= I(G_{S,2l} - w; -1) - I(G_{S,2l} - N[w]; -1) \\ &= I(G_{S,l} \cup C_3; -1) - I((G - S) \cup 2K_2; -1) \\ &= -2I(G_{S,l}; -1) - I(G - S; -1) \end{aligned}$$

as required. □

We see from parts (a) and (b) of the lemma that the first and second lateral joins "move up a level", i.e. the resulting graphs are one and two levels higher, respectively, than the original graph. Unfortunately, in an analogous third lateral join the decycling number would increase not by three but by four, as the three vertices added during the operations would form yet another triangle.

The forests $G - S$ for the graphs we consider all have a particular form: they are unions of K_2's and *double stars.* A double star D_n is a star $K_{1,n}$

whose edges have all been subdivided once. (Alternatively, a double star D_n is a set of n paths on 2 edges all sharing a common endvertex.) We note the following.

Lemma 2.7. *If D_n is a double star, then $I(D_n; -1) = (-1)^n$.*

Proof. Apply equation (1) with v equal to the center vertex of the double star. Since $D_n - v = nK_2$ and $D_n - N[v] = nK_1$, we have

$$I(D_n; -1) = I(nK_2; -1) - I(nK_1; -1)^n = (-1)^n - 0^n = (-1)^n.$$

□

We are now ready to complete level 4. Below we have pictured two level three graphs, one called $G^{3,-7}$ and the other called $G^{3,-6}$, both named for their level and q-value: clearly both are level 3 graphs and it is easy to verify that $q = -7$ for $G^{3,-7}$ and $q = -6$ for $G^{3,-6}$. (We mention in passing that both of these can be described using lateral joins: $G^{3,-7} = (2C_3)_{S,l}$ and $G^{3,-6} = C_3 \cup ((C_3)_{S,l})$, where S is any decycling set of the given graphs.) For both of these graphs we take S to consist of one vertex each of the lower (degree 2) vertices of the three triangles.

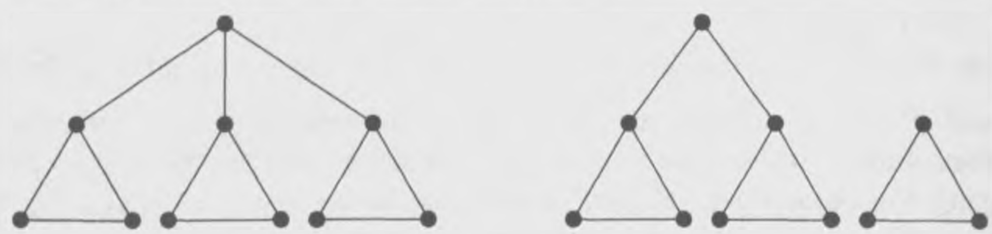

FIGURE 2. The level 3 graphs $G^{3,-7}$ and $G^{3,-6}$.

Theorem 2.8. *Let $G^{3,-8} = 3C_3$, and $G^{3,-7}$ and $G^{3,-6}$ defined as above. Then*

$$I((G^{3,-8})_{S,l}; -1) = 15$$
$$I((G^{3,-7})_{S,l}; -1) = 13$$
$$I((G^{3,-6})_{S,l}; -1) = 11$$

and all three graphs are level 4 graphs.

Proof. For illustration, the $q = 13$ graph is pictured in Figure 3 below.

That all three graphs are level four graphs is clear. Since $G^{3,-8} - S = 3K_2$, by Lemma 2.6 we have

$$I((G^{3,-8})_{S,l}; -1) = -2I(3C_3; -1) + I(3K_2; -1) = (-2)^4 + (-1)^3 = 15.$$

For the $q = 13$ result, since we have $G^{3,-7} - S = D_3$, where D_3 is a double star, using Lemmas 2.6 and 2.7 we have

$$I((G^{3,-7})_{S,l}; -1) = -2I(G^{3,-7}; -1) + I(D_3; -1) = (-2)(-7) + (-1)^3 = 13.$$

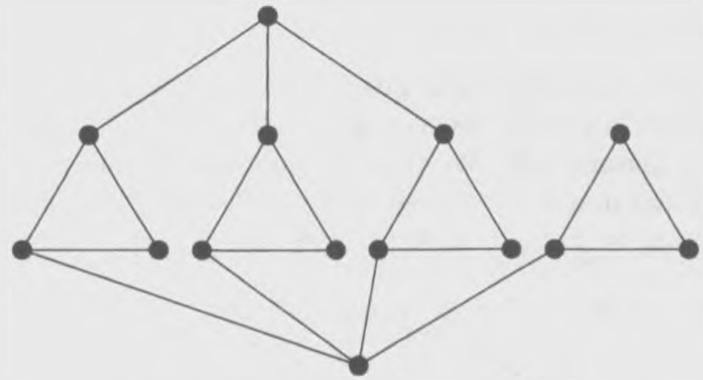

FIGURE 3. A level 4 graph G with $I(G;-1) = 13$.

And finally, since we have have $G^{3,-6} - S = D_2 \cup K_2$, where D_3 is a double star, using Lemmas 2.6 and 2.7 we have

$$\begin{aligned} I((G^{3,-6})_{S,l};-1) &= -2I(G^{3,-6};-1) + I(D_2 \cup K_2;-1) \\ &= (-2)(-6) + (-1)^2(-1) = 11. \end{aligned}$$

□

3. Level 5

With the machinery of the previous section in place, almost all of level 5 can be completed in the same way that level 4 was completed. As before, by Lemma 2.3 we need only consider the odd q, and of those only the odds larger (in absolute value) than 17. Ignoring signs, we need to produce level 5 graphs for q-values in $\{31, 29, 27, 25, 23, 21, 19\}$. With the exception of the $q = 19$ graph, all of these graphs can be constructed using first or second lateral joins on the $\mathcal{G}_4$ graphs.

In the theorem below it is understood that the decycling set of $G^{k,q} \cup C_3$ is the previously defined decycling set of $G^{k,q}$ together with any vertex of the C_3.

Theorem 3.1. *Let $G^{4,15}$ be the $q = 15$ graph defined in Theorem 2.8, with decycling set S equal to one vertex each of the degree 2 vertices of the four triangles. Then*

$$\begin{aligned} I((4C_4)_{S,l};-1) &= -31 \\ I((G^{4,15})_{S,l};-1) &= -29 \\ I((G^{3,-7} \cup C_3)_{S,l};-1) &= -27 \\ I((G^{3,-7})_{S,2l};-1) &= -25 \\ I(((C_3)_{S,l} \cup 2C_3)_{S,l};-1) &= -23 \\ I((G^{3,-6})_{S,2l};-1) &= -21 \end{aligned}$$

and all graphs are level 5 graphs.

Proof. The fact that all graphs given are level 5 is straightforward to confirm. As in the previous theorem the calculation of the q-values follows directly from Lemma 2.6. We first do the calculations for $-31, -29, -27$, and -23. By Lemma 2.6(c), since $(4C_4) - S = 4K_2$ we have

$$I((4C_4)_{S,l}; -1) = -2I(4C_4; -1) + I(4K_2; -1) = (-2)(-2)^4 + (-1)^4 = -31.$$

Since $(G^{4,15}) - S$ is the double star D_4, then

$$I((G^{4,15})_{S,l}; -1) = -2I(G^{4,15}; -1) + I(D_4; -1) = (-2)(15) + (-1)^4 = -29.$$

Since $(G^{3,-7} \cup C_3) - S = D_3 \cup K_2$, where D_3 is a double star, then

$$\begin{aligned} I((G^{3,-7} \cup C_3)_{S,l}; -1) &= -2I(G^{3,-7} \cup C_3; -1) + I(D_3 \cup K_2; -1) \\ &= (-2)(-7)(-2) + (-1)^3(-1) = -27. \end{aligned}$$

And since $(C_3)_{S,l} \cup 2C_3) - S = D_2 \cup 2K_2$, where D_2 is a double star, and $C_3 - S = K_2$, then

$$\begin{aligned} I((C_3)_{S,l} \cup 2C_3)_{S,l}; -1) &= -2I((C_3)_{S,l} \cup 2C_3); -1) + I(D_2 \cup 2K_2; -1) \\ &= (-2)^3 I((C_3)_{S,l}; -1) + (-1)^2(-1)^2 \\ &= (-2)^3(-2I(C_3; -1) + I(K_2; -1)) + 1 \\ &= (-2)^3((-2)^2 + (-1)) + 1 = -23. \end{aligned}$$

The graphs for q-values -25 and -21 are obtained via second lateral joins of the graphs used in Theorem 2.8. By Lemma 2.6(d), since $G^{3,-7} - S = D_3$, we have

$$\begin{aligned} I((G^{3,-7})_{S,2l}; -1) &= -2I((G^{3,-7})_{S,l}; -1) - I(G^{3,-7} - S; -1) \\ &= (-2)(13) - (-1)^3 = -25 \end{aligned}$$

and since $G^{3,-6} - S = D_2 \cup K_2$, we have

$$\begin{aligned} I((G^{3,-6})_{S,2l}; -1) &= -2I((G^{3,-6})_{S,l}; -1) - I(G^{3,-6} - S; -1) \\ &= (-2)(11) - (-1)^2(-1) = -21. \end{aligned}$$

This completes the proof. □

Although we have not yet found a graph for $q = \pm 19$ on level 5 that can be formed using the lateral join operation, the graph pictured in Figure 4 below, which we name $G^{5,-19}$, does complete the level and does so by using very similar ideas. In fact it is not difficult to verify that $G^{5,-19}$ is in fact the graph $G^{5,-21}$ with a single edge removed.

To conclude the paper we verify the q-value of the graph G shown. To do so, we call the bottom two vertices v and w. When w is deleted the resulting graph is the union of $2C_3$ and a connected graph; for reference, call that connected graph H. Note that $H - v = G^{3,-6}$ and that $H - N[v] = D_2 \cup K_2$,

FIGURE 4. A level 5 graph G with $I(G;-1)=-19$.

and also note that $G-w=G^{3,-6}\cup 2C_3$ and $G-N[w]=D_2\cup 3K_2$. Using equation (1) on vertex w we have

$$\begin{aligned}I(G;-1)&=I(G-w;-1)-I(G-N[w];-1)\\&=I(H\cup 2C_3;-1)-I(D_2\cup 3K_2;-1)\\&=(-2)^2I(H;-1)-(-1)^2(-1)^3\end{aligned}$$

and now using equation (1) again on v (with respect to graph H) we obtain

$$\begin{aligned}&=4\left(I(H-v;-1)-I(H-N[v];-1)\right)+1\\&=4\left(I(G^{3,-6};-1)-I(D_2\cup K_2;-1)\right.\\&=4((-6)-(-1)^2(-1))+1\\&=-19.\end{aligned}$$

REFERENCES

[1] A. Engström, Upper bounds on the Witten index for supersymmetric lattice models by discrete Morse theory, *European J. Combin.* 30 (2009), 429–438.

[2] I. Gutman and F. Harary, Generalizations of the Matching Polynomial, *Utilitas Math.* 24 (1983), 97–106.

[3] V. E. Levit and E. Mandrescu, The independence polynomial of a graph, a survey, *Proceedings of the 1st International Conference on Algebraic Informatics*, Aristotle Univ. Thessaloniki, Thessaloniki, 2005, pp. 233–254.

[4] V. E. Levit and E. Mandrescu, A simple proof of an inequality connecting the alternating number of independent sets and the decycling number, *Discrete Math.* 311 (2011), 1204–1206.

[5] V. E. Levit and E. Mandrescu, The cyclomatic number of a graph and its independence polynomial at −1, *Graphs Combin.* 29 (2013), 259–273.

[6] D. West, *Introduction to Graph Theory* (2nd edition). Prentice Hall, Upper Saddle River, NJ, 2001.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SETON HALL UNIVERSITY, SOUTH ORANGE, NJ

E-mail address: nathan.kahl@shu.edu