**5. Metric Spaces**

*2. Continuity in Metric Spaces*

Last time we introduced the concept of a metric space and we considered several examples:

* We looked at and imbued it with a variety of different metrics to create the metric spaces and in general .
* Similarly, we looked at the space of continuous functions C[a.b] and added different metrics to it to create spaces such as , and in general
* We also introduced a number of named inequalities, in particular the Cauchy-Schwartz Inequality (vector and integral version), Minkowski’s inequality (vector and integral version), and Hoelder’s inequality, which are used to establish the triangle inequality for different metrics

Metric spaces with infinite dimensions are possible as well: consider the space of all infinite sequences such that . Define a metric on this space via

This metric space is denoted by . Note: the properties of being a metric follow readily from by taking limits.

Now that we know several metric spaces, we can define familiar concepts on them:

**Definition 2.1: Continuity in a Metric Space**Suppose, where and are metric spaces. Then is continuous at a point if given any there exists a such that whenever then   
 is continuous on if it is continuous for all .

This is just our familiar concept of continuity, wrapped in a more abstract setting. However, we can now check continuity for new (and pretty neat) “functions”.

**Example 2.2**: Let be defined as . Find a few examples for the function and determine if it is continuous.

So, the” function” takes continuous functions and maps them to their value at . For example:

To work out continuity, we have to use the various metrics: fix an element and start with any . We want to have that whenever . In other words, we want

whenever . But that means that we can take and we have proved continuity of this operator because

Here is another interesting example. As a matter of fact, all of these examples are totally interesting (I think) because we apply the concept of old-fashioned continuity (graph has no holes; can draw graph without lifting the pencil) and apply it to situations where the graphical interpretation of continuity does not make any sense at all, yet the definition of continuity is so similar to our old fashioned one.

**Example 2.3:** Consider the operator defined by mapping an element of to another element in by dropping the first element of the sequence :

That operator is called the (forward) shift operator. Is it continuous?

First, to get a feel for this operator, we got to look at a few examples:

Take . Then because . Then

We can repeatedly apply the operator to drop the first few elements of the input sequence. For example

We need to check continuity: in other words, we want to find an such that whenever . But with and that is equivalent to:

But that is clearly the case, because

and we can take .

Here is one more example about continuity:

**Example 2.4**: Take ,. Is continuous?

As usual, let us get a feel for this operator first:

Now that we kind of know the operator L, we see that by the Fundamental Theorem of Calculus, is a continuous (even differentiable) function on [0,2], so that L is well defined. As for continuity: If then

But this implies continuity by letting

**Definition 2.5: (Discrete Metric)**Let be some set. We can turn into a metric space by endowing it with the **discrete** **metric**

As the above definition shows, any set X can be turned into a metric space. However, that metric is not very interesting, since it renders any function as continuous:

**Proposition 2.6 (Discrete Metric Space and Continuous Functions)***Any* function that maps a discrete metric space to any metric space is continuous.

The prove of this proposition is left as an exercise.

**Definition 2.6: (Homeomorphism)**If is a 1-1 mapping from metric spaces onto such that both and are continuous, then is called a **homeomorphism** and and are called **homeomorphic**.

**Definition 2.7: (Isometry)**  
If is 1-1 a mapping from metric spaces onto such that, then is called an **isometric** mapping or **isometry**.

Metric spaces that are isometric have the same metric relations between their elements. They are considered the same metric space, for all intent and purposes.

**Exercise:**

1. Show that Hoelder’s inequality implies Cauchy Schwartz (vector version)
2. Show that Minkowski’s inequality (vector version) may fail for .It would be okay to come up with a concrete example.
3. Prove that the function is a metric on the natural numbers
4. Let and . Find in . Note: the answer is a single number
5. Define the vector spaces , , and for . Give it your best shot (and model your definitions after and , and ). You do not need to prove anything, just say what you think a good definition might be.
6. For what vector spaces does the abstract definition of continuity reduce to our familiar one defined in calculus
7. If we take by defining (again): , is that operator continuous? How about where
8. Let the space of all bounded sequences. Add a metric by defining . Then is a metric space. Define by setting . Is continuous?
9. Show that the real line and the interval are homeomorphic.
10. Is the map from example 2.4 1-1? How about onto? Is it a homeomorphism? How about the map from example 2.3? And how about the one from example 2.2?
11. Prove proposition 2.6: Any function that maps a discrete metric space to any metric space is continuous.
12. Find the (open and closed) unit ball for a discrete metric space X. Recall that the open unit ball is and the closed unit ball is
13. Let denote the unit circle in the plane: . Define a map by . Prove that is continuous and bijective, but is not a homeomorphism.
14. Let and be the following disks in the plane (i.e. in with the usual Euclidian metric):

Show that and are homeomorphic.

1. Show that a bijective isometry between metric spaces is also a homeomorphism
2. Show that and are isometric.