**Lebesgue Integration**

*Appendix: Compact Sets and Open Coverings*

The version of the Heine Borel theorem that we covered in Analysis 1 was slightly non-standard and optimized to our particular needs of IRA. We defined a compact set as follows:

**Definition**: *A set S of real numbers is called compact if every sequence in S has a subsequence that converges to an element again contained in S.*

Then we proved the characterization of compact subsets of the real numbers (and in fact of $R^{n}$) as follows:

***Theorem****: A subset of R is compact if and only if it is closed and bounded*

Finally, we proved what we called the Heine-Borel theorem as follows:

***Theorem (Heine-Borel)****: A set S of real numbers is compact if and only if every open cover of S can be reduced to a finite subcover.*

All of these results are correct, but they are not suitable for a more abstract setting. For example, defining compact sets by requiring that every sequence in S has a convergent subsequence requires, naturally, the concept of convergence, which in turn is dependent on the underlying space being a metric space, i.e. a space with the concept of “distance”. We want to place the concept of compactness into the most general setting possible. Thus, we only want to assume a space with the concept of open (and therefore closed) sets, nothing else. Such spaces are called topological spaces, and we will study them in more details later. For now, we want to change our (general) definition of compact sets (in an abstract) topological space) to:

**Definition**: A subset S of a topological space X is called compact if every open cover of S can be reduced to a finite subcover.

A **cover** of a set $S$ is simply a collection of sets $\left\{C\_{α}\right\}$ whose union covers $S$, i.e. $S⊂\bigcup\_{α}^{}C\_{α}$. An **open** cover is a cover $\left\{C\_{α}\right\}$, where all $C\_{α}$ are open sets, and a **subcover** is a subset of the original collection $\{C\_{α}\}$ whose union still covers the set $S$.

**Example**: Here are a few covers and subcovers to find:

1. Find an open cover of the interval $S=[0, 1]$.
2. Find a finite cover of $[0, 1]$, i.e. a collection of finitely many sets that cover $S$.
3. Find a cover of [0, 1] consisting of countably many open sets. Reduce that cover to contain finitely many sets only. Does every finite collection from these sets necessarily form a subcover?
4. Can you extract a finite subcover of $S$ from the countable subcover from the previous example?

Here are the answers:

1. A simple open cover of the interval [0, 1] is, for example, the open interval (-1, 2), or even $(-\infty ,\infty )$, because S is a subset of either of these sets.
2. A finite open cover of [0, 1] is, for example $\left(-1,\frac{3}{4}\right)∪(\frac{1}{4},2)$ but not the collection of sets $\left(-1,\frac{1}{2}\right)∪\left(\frac{1}{2},2\right)$ (why not)?
3. A countable open cover of [0, 1] is, for example, the collection of sets $\left\{(-\frac{1}{n},1+\frac{1}{n})\right\}\_{n=1}^{\infty }$ We can reduce this cover to a finite subcover by taking, for example, only the first three sets from that collection $\left(-1,2\right)∪\left(-\frac{1}{2},\frac{3}{2}\right)∪(-\frac{1}{3},\frac{4}{3})$. Note that this sub-collection is indeed another subcover, since in fact each set of this collection includes, or covers, $[0, 1]$. Thus, *any* subset from this cover would be a subcover of [0,1].

But this is not always the case: consider the open cover of [0, 1] consisting of $\left(\bigcup\_{n=4}^{\infty }\left(-1,\frac{1}{2}-\frac{1}{n}\right)\right)∪\left(\bigcup\_{n=4}^{\infty }\left(\frac{1}{2}+\frac{1}{n},2\right)\right)∪(\frac{1}{4},\frac{3}{4})$.It is perhaps not trivial to see that this collection forms an open cover of [0, 1] but with a little work it should become clear. Note that not *every* finite subcollection will cover the original set, for example $\left(-1,\frac{1}{2}-\frac{1}{4}\right)∪\left(\frac{1}{2}+\frac{1}{4},2\right)=\left(-1,\frac{1}{4}\right)∪(\frac{3}{4},2)$ does not cover [0, 1].

1. However, I *can* find a finite subcover of the original set, picking, for example the first sets from each collection together with the last: $\left(-1,\frac{1}{2}-\frac{1}{4}\right)∪\left(\frac{1}{2}+\frac{1}{4},2\right)∪\left(\frac{1}{4},\frac{3}{4}\right) $. Oops – that last subcover does actually *not* cover the set S, but it is easy now to find a finite collection that does – do it!

**Example**: Find an open cover of the open set (0, 1) that cannot be reduced to a finite subcover. Try the same for the closed interval [0, 1], if possible, i.e. find an open cover that cannot be reduced to a subcover.

Consider the collection of open intervals $\left\{\left(\frac{1}{n},1-\frac{1}{n}\right)\right\}\_{n=2}^{\infty }$. It forms an open cover of (0, 1) because $\bigcup\_{n=2}^{\infty }\left(\frac{1}{n},1-\frac{1}{n}\right)=(0,1)$. Now take any *finite* subcollection of $\left\{\left(\frac{1}{n},1-\frac{1}{n}\right)\right\}$. Let $N$ be the largest index of the finite subcollection. Then $\bigcup\_{n\leq N}^{}(\frac{1}{n},1-\frac{1}{n})=(\frac{1}{N},1-\frac{1}{N})$ which does not include – or cover – the original set (0,1). Thus, for the set (0,1) it is not true that every open cover can be reduced to a finite subcover.

Moving on to $S=[0, 1]$. Let’s start with the collection of open intervals $\left\{\left(\frac{1}{n},1-\frac{1}{n}\right)\right\}\_{n=2}^{\infty }$, just as before. But that ain’t no open covering in the first place (why not), so … bad example. So, let’s include two open sets $(-ϵ,ϵ)$ and $(1-ϵ, 1+ϵ)$ together with the collection $\left\{\left(\frac{1}{n},1-\frac{1}{n}\right)\right\}\_{n=2}^{\infty }$ to make it an open cover of S. However, now you *can* reduce it to a finite subcover: regardless how small $ϵ$ might be, there is an $N$ with $\frac{1}{N}<ϵ$. But then the finite collection $(-ϵ,ϵ)$, $(1-ϵ, 1+ϵ)$, $\left\{\left(\frac{1}{n},1-\frac{1}{n}\right)\right\}\_{n=2}^{N}$ will be an open subcover of [0, 1]. Too bad … Let’s try a similar collection $\left\{\left(-\frac{1}{n},1+\frac{1}{n}\right)\right\}\_{n=2}^{\infty }$. That is indeed an open cover of the closed interval [0, 1], but so is each single set in the collection. Thus, this open cover certainly has many finite subcovers …. another bad example. How about if we defined the set $U\_{j,n}\left(r\right)=\{x:\left|x-\frac{j}{n}\right|<r\}$. Each set $U\_{j,n}(r)$ is open, and the collection $\left\{U\_{j,n}(ϵ)\right\}\_{j=0}^{\infty }$ covers the interval [0, 1] for every $n$ and every $ϵ>\frac{1}{n}$. So … am I lucky this time or can I reduce this open cover again to a finite subcover?

It turns out, that try as I might, it is not possible to find an open cover of the interval [0, 1] that cannot be reduced to a finite subcover, because of the following theorem, traditionally called the Heine Borel theorem:

**Theorem (Heine Borel)**

A subset of $R^{n}$ is compact if and only if it is closed and bounded.

**Proof:** The Heine Borel theorem is true for$R^{n}$but we will only prove it for$R$.There are two directions to prove (one easy the other one hard). The easy one first:

If a subset of $R$ is compact then it is closed and bounded. So, start with a compact subset S of R. Take an open cover of S of sets $U\_{x}\left(1\right)=\{y:\left|y-x\right|<1$ for all $x\in S$. Since S is compact, the open cover can be reduced to a finite subcover, say of N sets. Then $S⊂[x\_{min}-1,x\_{max}+1]$, with $x\_{min}=min⁡\{x\_{1},x\_{2},.., x\_{N}\}$ and $x\_{max}=max⁡\{x\_{1},x\_{2},.., x\_{N}\}$ so in particular S must be bounded. Next, we need to show that if S is a set for which every open cover has a finite subcover, then S must be closed.

**Remaining proof still to complete - TBD**

**Corollary:** If $S$ is a compact set in $R$, then $S$ contains its inf and sup, i.e. the inf and sup turn out to be min and max.

Proof: This would be a good HW assignment

So, we now have set the record straight: a compact set in a topological space is a set for which every open cover can be reduced to a finite subcover, and in $R^{n}$ the compact sets are precisely the closed and bounded sets.

**Exercises:**

* + 1. If $S$ is a compact set in $R$, then $S$ contains its inf and sup
		2. Find an open cover of the interval (a, b) that can be reduced to a finite subcover. Does this create a problem with the Heine Borel theorem, since (a, b) is not compact?
		3. Find an open cover of the interval (a, b) that cannot be reduced to a finite subcover.
		4. Consider the interval $S= [0, \infty )$ and the collection $U\_{n}^{j}=\{x:\left|x-j\right|<\frac{1}{n}\}$ for all $j$ and for all $n$. Show that the $U\_{n}^{j}$ cover the set S but you cannot reduce it to any finite subcover. What is your conclusion about the interval $S$, then?