**Lebesgue Integration**

*Part 1: Lebesgue Outer Measure*

Our next topic will be theoretical again and firmly rooted in (abstract) mathematics. We’ll start by quickly reviewing the Riemann integral and pointing out some problems with it. Our goal will be to eventually define a “better integral” that is more general yet agrees with the Riemann Integral whenever that one applies. To get to our goal, we will first extend the notion of “length” so that we can measure the size of any sets, even those different from intervals. Once that is done, we can define a new type of integral in a later section

Recall that the Riemann Integral was defined in various steps. And no, the definition has nothing to do with where is an antiderivative of (that one, while true if is bounded and piecewise continuous, is the “Integral Evaluation” theorem or Fundamental Theorem of Calculus) and it also not defined as the (net) area under the curve from to (that one is the geometric interpretation of *not* its definition). Rather it is:

**Definition (Riemann Integral)**

Let be a bounded function on the interval and define the lower and upper integrals, respectively, as

where and is called the lower sum of , and and is called the upper sum of . Then is Riemann integrable if and we write

We then managed to establish that a bounded function is Riemann integrable if and only if the function has at most countably many points of discontinuity, and we followed this up with a few techniques of integration.

The Riemann integral is certainly useful (and complicated) enough, but it does have a few limitations and oddities:

1. What happens when you change the value of a Riemann integrable function at a single point?

*Well, it depends. If you change the value of a single point to infinity, the new function is no longer Riemann integrable. If you change it to another bounded value, the value of the integral will not change. Prove that!*

1. Is it true that a function that is constant except at countably many points is Riemann integrable?

*No. It is true that a function that is continuous except at countable many points is R integrable, but there is a function that is constant except for countably many points that is not R integrable. What function might that be?*

1. What is the difference between Riemann integrable functions and bounded continuous functions?

*Not much, as it turns out. What, exactly, is the difference?*

1. Can you take a Riemann integral over anything else but an interval?

*No, not really. Our notion of R integral depends on partitions, which depends on intervals (or at best unions of intervals).*

1. Could you define a Riemann integral of a function whose domain is not R?

*Well, not really. You could extend it to but that’s about it. The definition depends on the structure of the domain, R or possibly , because of the nature of a partition.*

1. There is a sequence of Riemann integrable functions  that converges to a function  that is not Riemann integrable

*Take a sequence of functions that converge to the Dirichlet function. Details, please!*

1. There is a sequence of Riemann integrable functions  that converges to another Riemann integrable function f but the corresponding sequence of Riemann integrals of  does not converge to the integral of .

*Try and work out the details.*

We therefore want to define another concept of integration that is more general than the Riemann integral, yet retains the "good" properties of that integral.

One of the limitations of the Riemann integral is that it is based on the concept of an "interval", or rather on the length of subintervals . We therefore need to find a generalization of the "length" concept of a set in the real line. That new "length" concept, which we will call "measure", should satisfy two key conditions:

1. The new "measure" concept should be applicable to intervals, unions of intervals, and to more general sets (such as a Cantor set). Ideally, it should be defined for all sets.
2. The new "measure" concept should share as many properties as possible with the standard length of an interval, such as:

* the 'measure' of any set should be non-negative
* the 'measure' of an interval should be the length of that interval, but we should be able to ‘measure’ sets other than intervals as well
* the 'measure' of countably many disjoint sets should be the sum of the 'measures' of the individual sets

To define this basic concept, which we will end up calling Lebesgue Measure, we follow a two-stage strategy:

Stage One:

We will define a concept extending length that is defined for all sets (to satisfy condition 1 above)

Stage Two:

We will restrict the ‘stage one’ concept so that it looks as close as possible to the standard length concept (to satisfy condition 2 above)

The stage-one concept is called Lebesgue outer measure, defined as follows:

**Definition (Lebesgue Outer Measure)**

If  is any subset of , define the (Lebesgue) outer measure of  as:

where the infimum is taken over all collections of open intervals that cover A, i.e. such that , and is the standard length of the interval .

This is quite a definition. First, you need to find an “open cover” of a set by intervals , which simply means that the union of the cover the set . That union, by the way, could involve finitely or countably many intervals. Then you need to find the (possibly infinite) sum of the lengths of all these intervals. Finally, you need to find the smallest such sum, taking all possible coverings of A into account. Still confused? I don’t blame you, so it’s time for some examples:

**Example**: Find

Well, we need a few open covers of the empty set. Clearly each set (-1/n, 1/n) covers the empty set for any . Thus, for any , which means that .

**Example**: Find

The answer, to take a (not so) wild guess, should be , don’t you agree, but it is surprisingly difficult to prove it and we must summon some relatively big guns.

First, we cover in the open interval . Thus, we get for all n, so that . That was easy – now we’ve got to show that also, which would finish the proof.

So, cover the interval by a *finite* collection of open intervals , . Reorder these open intervals so that contains , contains , contains , and so on, until includes .

http://www.mathcs.org/analysis/reals/integ/graphics/outmsr1.gif

Then

Thus, for any finite cover of we have that . Now take *any* open cover of the closed, bounded interval [a, b]. Since that interval is compact, the cover can be reduced to a finite sub-cover by the Heine-Borel theorem, for which our previous estimate applies. Thus, for any open cover we have that

Thus, we have proved that , as conjectured.

By the way, the “big guns” mentioned at the beginning of the example is the Heine-Borel Theorem. The version of that theorem that we covered in Analysis 1 was slightly non-standard so I think an excursion to “open coverings” might be worthwhile – see the next sections for details.

**Example**: Find the outer measure of all rational numbers inside [0, 1]

Let be the set of all rational numbers in [0, 1]. Then is countable, so that . For each define a set . Then clearly the union of the will cover the set R, so that . Since was arbitrary, it follows that .

**Theorem** (**Properties of Outer Measure)**

The (Lebesgue) outer measure has the following properties:

1. The outer measure m\* is a non-negative set function whose domain is , i.e. the power set of **R**. In other words, every subset of R has non-negative outer measure
2. The outer measure of an interval is its length.
3. The outer measure is subadditive, i.e.
4. The outer measure is *countably* subadditive, i.e. if  is a countable collection of sets, then

For the proof of this theorem, please check <http://www.mathcs.org/analysis/reals/integ/proofs/propout.html>

**Exercises:**

* + 1. Prove statement 1 on oddities of the Riemann integral, i.e. that value of an integral does not change if you change the value of the function at a single point.
    2. Identify the function mentioned in statement 2
    3. Provide the details for statement 6 on the oddities of the Riemann integral
    4. Prove statement 7 on the oddities of the Riemann integral
    5. If and are two sets with , then
    6. Find , i.e. the outer measure of a single point
    7. Find , i.e. the outer measure of the open interval (a, b).*Hint: use the fact that*
    8. Show that *Hint: use the fact that outer measure is monotone*
    9. Find the outer measure of an infinite interval
    10. Find the outer measure of a countable set of points in
    11. What is the point of defining the Lebesgue outer measure. Why don’t we just use the length of a set instead of defining this strange and complicated outer measure. In other words, what is the advantage of outer measure over, say, length?
    12. Define the length of an interval from a to b as b – a. Is length subadditive or additive (a function is additive if for A, B disjoint, and subadditive if for any A, B).
    13. Define a set function (cm stands for “counting measure”) to be equal to the number of elements in , if is finite, or infinity. What is the domain and range of ? Is subadditive or additive? How about countably subadditive or additive?