**Fourier Series**

*Part 2: Convergence Issues*

So far we have proved the following theorem:

**Theorem**: If for then

In the previous segment we used these formulas to compute a number of Fourier series. In this segment we want to switch to a more theoretical discussion and we want to figure out the convergence behavior of a Fourier series. There are two parts to this question:

1. Find out for which functions a Fourier series converges
2. Determine for which functions a Fourier series converges to the original function

Let’s check the first question first (of course, or I would not have called question 1): Let’s start with some function :

* if , does converge
* if , does converge

The series are, on one level, just some infinite series, so we could apply the various convergence tests we know. Let’s start with the divergence test:

**Proposition 1:** If is a function (i.e. f is differentiable and f’ is continuous), then the divergence test fails. That means, of course, that and , and tells us nothing about convergence of the series.

To show this, let’s use integration by parts on the coefficients . Let and . Then and so that:

Since is continuous on a closed, bounded interval, it is bounded by, say, . But then

so that . But then for all x because , so that the divergence test fails.

The proof for the other series is similar – but not quite: make sure to try it!

Based on this proof it is now easy to show that:

**Proposition 2:** Suppose is a function (i.e. is twice differentiable and is continuous). Then the Fourier series with , and converges uniformly.

Note that this theorem will be easy to prove but it is not satisfying: for one thing, it does not show whether the series converges *to the original function*, and for another we know from our examples that the Fourier series for a function seems to converge (to the original function) for functions that are not even continuous, while this theorem assumes is twice continuously differentiable. But, it is a start, so let’s go prove it.

**Proof**: We already know that

Integrate by parts again, with , , and, consequently, , :

Therefore:

for some constant K, so that converges by the comparision test with , a convergent p-series with , and the convergence is uniform by Weierstrass Convergence theorem.

The proof that converges is similar. But then the original Fourier series must converge.

The above theorem is nice and easy to prove, but using Abel’s convergence test for series we can do slightly better:

**Proposition 3**: If is a function (i.e. f is differentiable and f’ is continuous), then the Fourier series for converges.

Apply Abel’s convergence test (see IRA 4.2 for details). The details are left as an exercise.

But we can do a little better still. Here is a ‘sharp’ result for the convergence of a Fourier series. The proof is a little detailed so we are going to skip it.

**Theorem: (Fourier Series)**

Suppose is a piecewise function, i.e. consists of finitely many parts, each of which is continuously differentiable, and all discontinuities are jump discontinuities, then the Fourier series of converges to at all points where is continuous, and to at each jump.

**Example**: Check the convergence behavior of the Fourier series of

Pay special attention to the limit of the Fourier series at the jump at

In the previous segment we worked out the Fourier series for this function to be . Clearly , so that the series does converge at least at the jump as advertised. And it seems to converge to elsewhere as well, judging by several graphs of the N-th partial Fourier series with , , and , respectively.

  

Note how the Fourier series ‘overshoots’ near the jump but then quickly dampens down to hug the original function closely. That behavior is typical for Fourier series and has a number of interesting consequences in the physics of signal transmission.

Anyway, we have achieved our goal: the Fourier Theorem above tells us for which functions the Fourier Series converges back to the original function.

We do have one pesky limitation left: each of the function we considered so far was defined on the interval . While we can extend the function by periodicity to all real numbers and the Fourier series would converge to that extended periodic function for all , what about functions defined on an arbitray interval [a, b]? We will check that out in our next segment.

**Exercises:**

1. Prove the second part of Proposition 1
2. Prove the second part of Proposition 2
3. Prove Proposition 3. Hint: Prove and then use the fact that together with Abel’s convergence test.
4. Use the Fourier series for on to show that (we already knew that by looking at the Taylor series for but our reasoning is different this time around)
5. Use the Fourier series for on to show that
6. Show that
7. Check the Fourier Theorem for , especially at the jump . Verify the convergence behavior graphically, using Mathematica.