**Power / Taylor Series**

*Part 5: Taylor’s Theorem*

We have a few basic power series at our disposal and we have learned some tricks to find more. We also learned about Taylor series and that not every power series is a Taylor series. Finally, we came up with an example of a function that is , does have a Taylor series expansion, but whose Taylor series does not converge to the original function. That’s the bad news. As for the good news, we will come up with criteria that will regulate when a Taylor series converges back to its original function in this segment (the *last* on power series).

**Theorem (Taylor)**

Suppose is a function. Then

where

In particular, if is a function then the Taylor series for the function converges to the function for all if and only if for all .

This looks like a hard theorem to prove, but the statement holds for all integers so we go for an induction argument:

Check :

Assume case n is true: where

We will use *Integration by Parts* on : let and . Then and so that

But that means that, using our induction hypothesis

which finishes the (pretty nifty!) induction step.

The second part of the theorem is much easier: Since , the n-th Taylor polynomial converges to if`and only if the reminder goes to zero.

**Corollary**: Every function can be approximated by a polynomial of degree

**Example**: Approximate by a second degree polynomial near the origin.

We know that



**Example**: Show that  is equal to its Taylor series for all x

We already know that and to show that we need to verify that

But

because the exponential function is increasing, and since for all we have as required

Note: Only at *this* stage, after all this work, are we *certain* that It turns out that computing a Taylor series for a given function is actually the easy part, the hard part is to show that the reminder goes to zero, i.e. to show that the Taylor series really does represent the original function. Therefore it is advantageous to have other forms of the reminder that are sometimes easier to estimate:

**Theorem: (Lagrange Remainder Formula)**

Suppose is a function. Then the function has a Taylor polynomial of degree n with remainder

for some between and . As with Taylor’s reminder, the series converges to iff as

Proof: We need to show that the Taylor’s Remainder can be transformed into the Lagrange form . Recall the MVT for integrals:

for some d between . With and we have that

which is what we wanted to show.

**Example**: Show that the Taylor series for converges to the function.

We need to show that the reminder goes to zero. Using the Lagrange form of the reminder this is pretty simple:

as for any , since and

Note that (real-valued) functions that have a Taylor series expansion are called **real analytic**. **Real analytic** functions are the truly nice functions and you can deduce many properties by looking at the series representation of the function. The term ‘real analytic’ is related to the definition of an analytic function in complex analysis; (complex) analytic functions are much easier to understand and it turns out that a function of a complex variable that is once (complex) differentiable is necessarily analytic, i.e. it is infinitely often differentiable *and* it has a Taylor series expansion. That most definitely is not true for real valued functions: there are functions that are once differentiable but not twice, twice but not thrice, etc. And even infinitely often differentiable functions are not necessary real analytic. Thus, the concept of complex derivative is much more restrictive for functions of one complex variable but it leads to a much ‘cleaner’ theory.

To illustrate how useful Taylor series can be, consider the following example:

**Example**: Find

We could of course use l’Hospital’s rule multiple times, but it is annoying to compute higher and higher order derivatives for the top function. However, if we used the series expansion of the problem turns out to be super easy:

**The Most Beautiful formula in Mathematics:** With , prove Euler’s Equation . Use it to prove what many call "*the most remarkable formula in mathematics*" , known as Euler’s Formula. It is remarkable because it combines the most basic and important yet seemingly different and unrelated numbers , and 0 in one easy equation.

I leave the proof to you, but I would start by using the Taylor series for , substitute and then use the fact that
, , and , etc.

**Exercises**

1. Find the first three nonzero terms in the Taylor series for  on  and estimate the error
2. Find a polynomial approximation for  on  accurate to
3. Prove the theorem we stated in our last segment, i.e. that for any polynomial.
4. Show that  is equal to its Taylor series for all