**Power / Taylor Series**

*Part 1: By Finding , Differentiating, or Integrating*

We know that if a power series has a radius of convergence then:

* The series converges uniformly and absolutely for all to a continuous function
* The functions is differentiable and
* The function is integrable and

Moreover, if that series represents a (known) function f(x) then we have:

* so that
* so that
* so that
* so that
* and so on

Thus, we can see (by induction) that or equivalently:

Based on this theory we can deduce (at least) three methods for finding power series:

1. Compute the n-th derivative of at and divide by :
2. Start with a known series and take its derivative
3. Start with a known series and integrate

Method 1 works well if there is an easy pattern to the derivatives of a function. The other two methods depend on knowing just the right series.

Examples:

1. Assuming that has a power series expansion centered at , find it and compute its radius of convergence.
2. We know the series expansion of the function . Based on that, find the series for
3. Assuming that there is one, find the power series for

For number 1, we can easily compute the n-th derivative of at . From there it is easy to figure out the coefficients for the series. If you work out the radius of convergence, you’ll find

We have so that

To find the radius of convergence, we apply the ratio test: the series converges if so that

for all x. That means that the radius is infinity.

For number 2 we note that . Since we can replace the function on the left by its power series, we can differentiate it to get the power series of the function we are interested in.

For number 3 we note that . Again, we can replace the function on the left by its power series and then we can integrate to get the desired power series.

To find the constant C, plug in for example . You will find that

Exercises:

1. Show that the derivative of the exponential function is again the exponential function
2. Assuming that there is one, find the series for and
3. Verify that by computing the n-th derivative of
4. Find the derivatives of and
5. Prove that , i.e. the alternating harmonic series adds up to