Math 4515: Sample Problems for Final Exam

1. **Taylor Series**
   a) Find Taylor series centered at zero for *ln(1+x)* or *arctan(x)* or $\frac{x}{(1-x^2)2}$
   b) Find the Taylor series centered at x = 2 of $f(x) = 1/x$
   c) Prove Lagrange's version of the remainder for Taylor series

2. **Lebesgues Outer Measure and Measure**
   a) Let $A = \boldsymbol{Q} \cap [0,1]$. Let A_N be a finite collection of open intervals covering *A*. Then $\sum l(A_n) \geq 1$
   b) Prove that the outer Lebesgue measure of a countable subset of ***R*** is zero. Use it to find the Lebesgue outer measure of ***Q***.
   c) If *A* and *B* are two sets with $m^*(B) = 0$ then $m^*(A \cup B) = m^*(A)$
   d) Show that every set in ***R*** with outer measure zero is measurable
   e) Does every subset *A* of ***R*** have an outer measure? How about a measure?
   f) Given any set *A* and any $\varepsilon > 0$ there is an open set *O* such that $A \subset O$ and $m^*(O) \leq m^*(A) + \varepsilon$
   g) Given any set *A* there exists a $G_\delta$set *G* such that $A \subset\ G$ and $m^*(A) = m^*(G)$. Note that a set is a G_δ set if it is the countable intersection of open sets (δ for "Durchschnitt", German for intersection).

3. **Lebesgue Integration**
   a) Show that $\mathrm{X}_{A\cap B}(x) = \mathrm{X}_\mathrm{A}(x)\,\mathrm{X}_B(x)$
   b) Show that $\mathrm{X}_{A\cup B}(x) = \mathrm{X}_A(x) + \mathrm{X}_B(x) - \mathrm{X}_\mathrm{A}(x)\,\mathrm{X}_B(x)$
   c) Show that if *f* is bounded and integrable and $\mu(E) = 0$ then $\int_E\ f\,d\mu = 0$
   d) Show that every continuous function on *[a, b]* is Lebesgue integrable.

4. **Fourier Series**
   a) Find the Fourier series of the function $f(x) = \begin{cases} 0 & \text{if} -\pi \leq x \leq 0 \\ 2 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad 1 \leq x \leq \pi \end{cases}$
   b) Find the Fourier series of *f(x) = |x|* defined on $[\pi, \pi]$
   c) Does the sequence of functions $\{f_n(x)\} = \left\{\frac{1}{\sqrt{1+(nx)^2}}\right\}$ converges in mean square on the interval $(-\infty, \infty)$? Prove it.

5. Metric Spaces

   a) Show that for any metric space (X, ρ) we have $|\rho(x,z) - \rho(y,z)| \leq \rho(x,y)$
   b) Is *C[a,b]* with the max norm complete? How about in the C^2 norm? In each norm, find $\rho(\sin(x), \cos(x))$ on $[0,2\pi]$
   c) Some operator yet to be determined on some metric space. Is it 1-1, onto, or continuous?
   d) Suppose E_n is a sequence of sets of first category. Show that the countable union of E_n is also of first category.