Math 4515: Final Exam

This is a take-home exam, due on the day of the final. You can submit it in person or via email.
You must complete the exam on your own, but you may use any other resource as necessary.

1. Taylor Series

a) Find Taylor series centered at zero for $f(x) = \ln(1+x^2)$, including the radius of convergence.

2. Lebesgues Outer Measure and Measure

a) State the definition of a set being Lebesgue measurable. Use that definition and some basic properties of measure to show that every set in ***R*** with outer measure zero is measurable.

b) Show that given any set *A* and any $\varepsilon > 0$ there is an open set *O* such that $A \subset O$ and $m^*(O) \leq m^*(A) + \varepsilon$. Then use that result to show that for any set *A* there also exists a G_δ set *G* such that $A \subset G$ and $m^*(A) = m^*(G)$. Note that a set is a G_δ set if it is the countable intersection of open sets (δ for "Durchschnitt", German for intersection).

3. **Lebesgue Integration**

a) State the definition of a bounded function being Lebesgue integrable. Use that definition to show that if *f* is bounded and integrable and $\mu(E) = 0$ then $\int_E f\, d\mu = 0$

b) Show that every continuous function on *[a, b]* is Lebesgue integrable.

4. Fourier Series

a) Find the Fourier series of the function $f(x) = \begin{cases} 0 & \text{if} -\pi \leq x \leq 0 \\ 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{if} \quad 1 \leq x \leq \pi \end{cases}$. When done, verify your calculations by plotting the sum of the first, say, 50 terms using Maple.

b) Does the sequence $\{f_n(x)\} = \left\{\frac{1}{\sqrt{1+(nx)^2}}\right\}$ converges in mean square on $(-\infty, \infty)$? Prove it.

5. Metric Spaces

a) Consider l_2, i.e. the space of all sequences of numbers $\{a_n\}$ such that $\sum a_n^2 < \infty$ with the standard metric $\rho(a,b) = \sqrt{\sum_{j=1}^{\infty}\left(a_j - b_j\right)^2}$. Define a function $f: l_2 \to l_2$ such that $f(a_1, a_2, a_3, \ldots) = (a_2, a_4, a_6, \ldots)$. Is *f* 1-1? Is it onto? Is it continuous? Justify your answers.

b) Is *C[a,b]* with the max norm complete? How about in the C^2 norm? Justify your answer. In each norm, find $\rho(\sin(x), \cos(x))$ on $[0, 2\pi]$