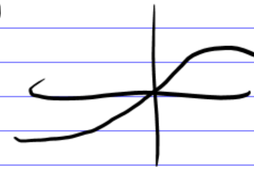
Panel 1

$f(x) = \arctan(x)$ $\quad f(0) = 0$ $\quad a_n = \frac{f^{(n)}(x_0)}{n!}$



$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1}$ $\quad f'(0) = 1$

$f''(x) = -\frac{2x}{(1+x^2)^2}$ $\quad f''(0) = 0$

$f'''(x) \ldots$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\arctan(x) = \int \frac{1}{1+x^2}\,dx = \int \sum (-1)^n x^{2n}\,dx = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

$$= x - \frac{1}{3}x^3 \qquad |x|<1$$

1

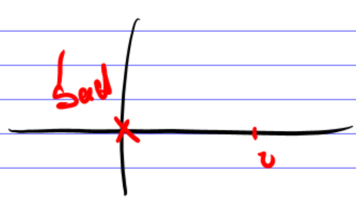
Panel 2

$$\frac{1}{x} = \frac{1}{-2-(x-2)} = \qquad (x-2)^n$$

$$= \frac{1}{-2}\,\frac{1}{1+\frac{x-2}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x-2}{2}\right)^n =$$

$$= -\frac{1}{2}\sum (-1)^n \frac{(x-2)^n}{2^n} =$$

$$= \sum (-1)^{n+1} \frac{(x-2)^n}{2^{n+1}} \qquad \forall\, |x-2|<2$$



2

Panel 3

2. Lebesgues Outer Measure and Measure

a) Let $A = \boldsymbol{Q} \cap [0,1]$. Let A_N be a finite collection of open intervals covering A. Then $\sum l(A_n) \geq 1$

b) Prove that the outer Lebesgue measure of a countable subset of $\boldsymbol{R}$ is zero. Use it to find the Lebesgue outer measure of $\boldsymbol{Q}$. 0

1. $\mu^*(\{x\}) = \emptyset$ 2. $\mu^*(\text{countable union}) \leq \sum \mu^*($

c) If A and B are two sets with $m^*(B) = 0$ then $m^*(A \cup B) = m^*(A)$

$A \subset (A \cup B)$

$\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$

→ d) Show that every set in $\boldsymbol{R}$ with outer measure zero is measurable Yes

e) Does every subset A of $\boldsymbol{R}$ have an outer measure? How about a measure? No

→ f) Given any set A and any $\varepsilon > 0$ there is an open set O such that $A \subset O$ and $m^*(O) \leq m^*(A) + \varepsilon$

→ g) Given any set A there exists a G_δ set G such that $A \subset G$ and $m^*(A) = m^*(G)$. Note that a set is a G_δ set if it is the countable intersection of open sets (δ for "Durchschnitt", German for intersection).

3

Panel 4

E is L-msble: $\mu^*(A) \geq \cancel{\mu^*(A \cap E)} + \mu^*(A \cap E^c) \quad \forall A$

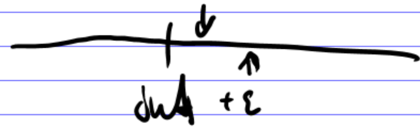
$\mu^*(E) = 0$, $A \cap E \subset E \Rightarrow \mu^*(A \cap E) = 0$

$A \cap E^c \subset A \Rightarrow$

4

Panel 5

$\mu^*(A) = \inf\{\sum \ell(A_j),\ A_j \text{ open intervals cover } A\}$



Given $\varepsilon > 0$ $\exists$ A_j open cover of A s.t.

$\sum \ell(A_j) \le \mu^*(A) + \varepsilon$

Let $O = \bigcup A_j$. $A \subset O$

$\Rightarrow \mu^*(O) \le \sum \mu^*(A_j) = \sum \ell(A_j) \le \mu^*(A) + \varepsilon$

5

Panel 6

O_n s.t. $A \subset O_n$, $\mu^*(O_n) \le \mu^*(A) + \frac{1}{n}$

$\bigcap_{n=1}^{\infty} O_n \to G_\delta\text{-set}$

$\mu^*(A) \le \mu^*(O) \le \mu^*(A) + \frac{1}{n} \quad \forall n$

$\Rightarrow$

6

Panel 7

$$\int_E f\,d\mu = 0 \quad \text{if } \mu(E)=0$$

Recall: If f is bdd, the L-int. is defined as

$$I^*(f) = \inf\left(\int_E s\,d\mu : s \text{ simple}, s \geq f\right) \;=$$

$$I_*(f) = \sup\left(\int_E s\,d\mu : s \text{ simple}, s \leq f\right)$$

$\chi_E(x)$

s simple: $s(x) = \sum_{j=1}^{n} a_j \chi_{A_j}(x)$ =

$$\Rightarrow \int_E s(x)\,d\mu = \sum a_j \mu(A_j \cap E)$$

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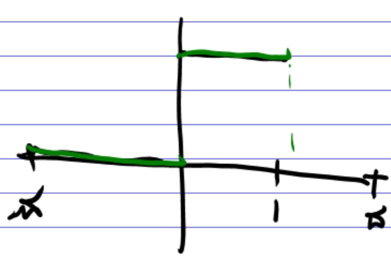
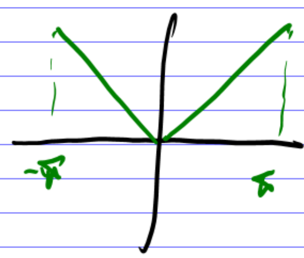
Panel 8





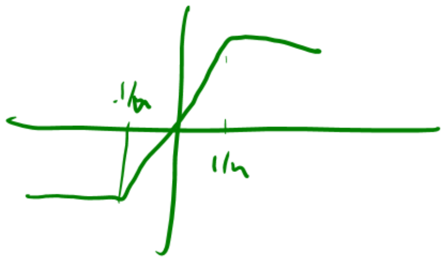
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\cdot\frac{1}{2}\pi^2 = \frac{\pi}{2}$$

$$a_j = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{-1+(-1)^n}{n^2}\cdot\frac{2}{\pi}$$

$$s_j = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\sin(nx)\,dx = 0$$

8

Panel 9



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