Panel 1

Complete Metric Spaces

Def: (X, ρ) a metric space, $\{x_n\}$ a sequence in X s.t. given $\varepsilon > 0$ $\exists N$ s.t. $\rho(x_n, x_m) < \varepsilon$ $\forall n,m \geq N$. Then $\{x_n\}$ is a fundamental, or Cauchy, sequence.

Thm: If $\{x_n\} \to x_0$ then x_n is Cauchy.

Proof: Given $\varepsilon > 0$ $\exists N$ s.t. $\rho(x_n, x_0) < \varepsilon$ $\forall n \geq N$

Given $\varepsilon > 0$ $\exists N$ s.t. $\rho(x_n, x_m) < \varepsilon$ $\forall n, m \geq N$

$\rho(x_n, x_m) \leq \rho(x_n, x_0) + \rho(x_0, x_m)$

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Panel 2

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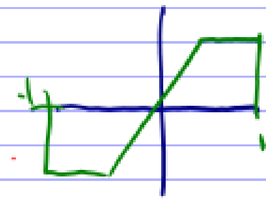
Panel 3

Is $C^0[a,b]$ complete? YES

Is $C^2[a,b]$ complete? NO



$\mathbb{R}$ complete? YES

$\mathbb{Q}$ complete? No

$\mathbb{N}$ complete? Yes

Consider $\mathbb{R}$ with $\rho(x,y)=|\arctan(y)-\arctan(x)|$.

$(\mathbb{R},\rho)$ is not complete.

$\{n\}$ $\Rightarrow$ $|\arctan(n+1)-\arctan(n)|$

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Panel 4

$$\frac{|\arctan(n+1)-\arctan(n)|}{(n+1)-n} = f'(c) = \frac{1}{1+s^2}, \quad s\in[n,n+1]$$

$$\Rightarrow \lim_{n\to\infty} |\arctan(n+1)-\arctan(n)| = 0 \quad \left(\frac{1}{1+s^2}\to 0 \text{ as } s\to\infty\right)$$

$$\Rightarrow \lim_{n\to\infty} |\arctan(n+2)-\arctan(n)| = 0$$

$$\Rightarrow |\arctan(n)-\arctan(m)| < \varepsilon$$

$\{n\}$ is Cauchy in ρ-metric. But it does not conv!

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Panel 5

Can any function serve to define a new metric on $\mathbb{R}$, i.e. is $(\mathbb{R}, \rho)$, $\rho(x,y) = |f(x) - f(y)|$, a metric space?

No: f needs to be 1-1

Need: st $\rho(x,y) = 0 \Rightarrow x = y$

i.e. $|f(x) - f(y)| = 0 \Rightarrow x = y$

HW Find $f(x)$ ($\neq$ arctan(x)) st. $(\mathbb{R}, |f(x) - f(y)|)$ is not complete!

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Panel 6

Nested Spheres:

Let $S[x_j, r_j] = \{x \in X : \rho(x, x_j) \leq r_j\}$ is closed sphere around x_j with radius r_j

A sequence of $\{S[x_j, r_j]\}_{j=1}^{\infty}$ is called nested if $S[x_{j+1}, r_{j+1}] \subset S[x_j, r_j]$ $\forall j$

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Panel 7

Thm (Nested Sphere Thm)

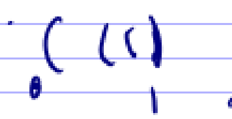
A metric space (X,ρ) is complete if and only if every sequence of nested, closed spheres $S[x_j, r_j]$ with $r_j \to 0$ has non-empty intersection.

Why closed spheres? Want nested spheres in $\mathbb{R}$ with $\bigcap S_n = \emptyset$.

$(-n,n)$ not nested

$(-\frac{1}{n},\frac{1}{n})$ nested but $\bigcap(-\frac{1}{n},\frac{1}{n}) = \{0\}$

$\bigcap$ is empty $\leftarrow$ $(0,\frac{1}{n})$ or $S[1-\frac{1}{n},\frac{1}{n}]$



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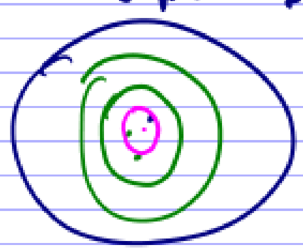
Panel 8

Proof: Suppose X is complete and $S[x_n,r_n]$ is a sequence of closed spheres with $r_n \to 0$.



$\{x_n\}$ the centers of these spheres.

$\{x_n\}$ is Cauchy, because $r_n \to 0$

x_n, x_m are in $S[x_n,r_n] \Rightarrow \rho(x_n,x_m) < r_n \to 0$

Since X is complete, $\lim x_n = x$.

But $x_j \in S_n$ $\forall n$ except $j = 1,2,3,\dots,n-1$.

$\Rightarrow x$ is a limit point of S_n, S_n is closed

$\Rightarrow x \in S_n$ $\forall n$ $\Rightarrow x \in \bigcap_{n=1}^{\infty} S_n$

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Panel 9

Now assume that every sequence of nested spheres with $r_n \to 0$ has non-empty intersection

look it up!

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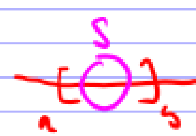
Panel 10

Recall: $A \subset X$ is dense in X if $\bar{A} = X$

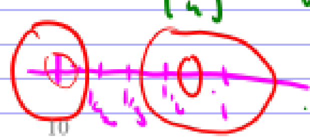
Def: $A \subset X$ is nowhere dense in X if it is dense in no open sphere at all, or:
every sphere in X contains another sphere S' s.t. $S' \cap A = \emptyset$, or:
$\text{comp}(\bar{A})$ is dense.



Ex: $\mathbb{N}$ is nowhere dense in $\mathbb{R}$

$[a,b]$ not nowhere dense in $\mathbb{R}$

$[\frac{1}{n}]$ nowhere dense in $\mathbb{R}$



?

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Panel 11

Baire's Thm: A complete metric space X cannot be written as a union of countably many nowhere dense sets.

Consequence: A complete metric space X without isolated points is uncountable.

[Proof] $\{x\}$ is nowhere dense. q.e.d.

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