Panel 1

Summary:

$\mathbb{R}^n_0$

$\mathbb{R}^n_1$

$\mathbb{R}^n_2$?

$C^0[a,b]$?

$C^2[a,b]$

ℓ_2

$\mathbb{R}^n_p$?

Schwarz inequ. ?

Hölder Inequ. ?

ext in ℓ_p ?

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Panel 2

Cauchy - Schwarz

Schwarz

Minkowski

Hölder

2

Panel 3

You define:

l_p

$C^p[a,b]$

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Panel 4

Last Time

Open nbhd of $x_0 \in (X,\rho)$: $O_r(x_0) = \{x \in X : \rho(x_0,x) < r\}$

Closure of $S \subset X$: $S \cup \{$bdry points$\}$

$\lim_{n\to\infty} x_n = x \iff \rho(x, x_n) \to 0$ as $n \to \infty$

A is dense in B if $A \subset B$ and
$\bar{A} \supset B$ (usually $\bar{A} = B$)

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Panel 5

Def: A metric space (X,ρ) is separable if it contains a countable dense subset, i.e. $S=\{s_1,s_2,\dots\}$ s.t. $\bar{S}=X$

Ex: $\mathbb{R}$ $(\bar{\mathbb{Q}}=\mathbb{R})$

$\mathbb{R}^n_{0,1,2}$ $(\mathbb{Q}^n)$ $\quad q_1, q_2, q_3, \dots q_n, \dots$ $\quad (x_1, x_2, x_n \dots$ D thing

l_2 (finite sequ. of rational points)

$C[a,b]$, $C^1[a,b]$ } polyn. with rational coeff!

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Panel 6

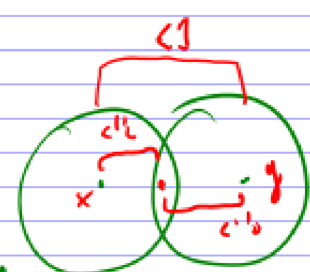
Ex: Consider the set m of all bounded sequences with $\rho(x,y)=\sup(|x_k-y_k|)$. Then m is a metric space. Is it separable? $\boxed{\text{NO!}}$

Define $S\subset m$, $S=\{\text{sequences of } 0\text{'s and } 1\text{'s}\}$

$\forall x\neq y\in S: \Rightarrow \rho(x,y)=1$

Consider $O_{1/2}(x)$, $x\in S$

$\Rightarrow$ The $O_{1/2}(x)$'s are disjoint



Suppose $\{x_1,x_2,\dots\}$ dense in m

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Panel 7

Each $O_{1/2}(x)$ must contain at least one x_j's

$\rightarrow$ $O_{1/2}(x)$'s are countable!

But what is card (S)? What is card of all seqn of 0's and 1's?

Thm card (S) = card $(\mathcal{P}(\mathbb{N}))$ = c

$\{1,2,3,4\ldots\} = \mathbb{N}$

$x = (0,1,1,0,0,1,0,0)$ $\quad f(x) = \{2,3,6\}$

f is onto + 1-1. Now: card (S) = c

$\{3,5,11,12,15,20,21,22,\ldots\} = f(0,0,0,0,0,0,1,0,1,0,1,1,0,0,1,0,0,0,0,1,1,1)$

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Panel 8

To summarize

$m = \{$ bdd. sequences $\}$, $\rho(x,y) = \sup(|x_k - y_k|)$

$S = \{s \in m : s = (s_1, s_2, \ldots)$ with $s_j = 1$ or $0\}$

For every $x \in S$ pick $O_{1/2}(x)$ $\Rightarrow$ card $(\{O_{1/2}(x)\})$ = card (S)

Showed that card (S) = c = card $(\mathcal{P}(\mathbb{N}))$

Finally: if there was a countable dense subset C of m then (dense) $\Rightarrow$ card $(\{O_{1/2}(x)\})$ = card (C) = countable

↯

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Panel 9

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Panel 10

Ex: X any set. Define $\rho : X \times X \to \mathbb{R}$

$$\rho(x,y) = \begin{cases} 0 & \text{if } \underline{x = y} \\ 1 & \text{if } x \neq y. \end{cases}$$

Is (X, ρ) a metric space? Is X separable?

What is $O_r(x)$?

triangle: $\rho(x,z) \leq \underbrace{\rho(x,y)}_{1} + \underbrace{\rho(y,z)}_{1}$ ✓ 1 or 1

$\rho(x,z)$: 1 ✓, 0 ✓

ρ is metric!

Any set $S \subset X$ is closed!. S is closed if it contains all its sdry points.

x is sdry point if $O_{1/2}(x)$

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Panel 11

$O_{1/2}(x)$ does not contain points in S and outside of S

$\Rightarrow$ there are no bdry points

$\Rightarrow$ S is closed trivially!

"isolated space"

If X was sep., then $\overline{S} = X$

but $\overline{S} = S$

Thus, if X was sep. $\Leftrightarrow$ X is countable

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Panel 12

Complete Metric Spaces

<u>Def</u>: (X, ρ) is metric space, $\{x_n\}$ a sequence in X.

If $\rho(x_n, x_m) < \varepsilon \quad \forall n, m \geq N$

then $\{x_n\}$ is called fundamental, i.e. Cauchy!

I.e. $\forall \varepsilon > 0 \ \exists N$ s.t. if $n, m \geq N$

$\Rightarrow \rho(x_n, x_m) < \varepsilon$

<u>Def</u>. (X, ρ) is <u>complete</u> if every Cauchy sequence converges to $x \in X$.

<u>Ex</u>: $\mathbb{R}$ is com.

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