Panel 1

Summary:

$\mathbb{R}^n_0$

$\mathbb{R}^n_1$ points: $(x_1, x_2, \dots, x_n)$, $x_j \in \mathbb{R}$ with

$\rho(x,y) = \sum_{k=1}^{n} |x_k - y_k|$

$\mathbb{R}^n_2$

$C[a,b]$

$C^2[a,b]$ points: $f: [a,b] \to \mathbb{R}$, cont., with

$\rho(f,g) = \left[\int_a^b (f-g)^2 \, dt \right]^{1/2}$

ℓ_2

$\mathbb{R}^n_p$ points $(x_1, x_2, \dots)$, s.t. $\sum_1^\infty x_j^2 < \infty$, $\rho(x,y) = \left(\sum_1^\infty (x_j - y_j)^2 \right)^{1/2}$

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Panel 2

Cauchy-Schwarz $\left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \left(\sum_1^n a_k^2 \right) \left(\sum_1^n b_k^2 \right)$

Schwarz

Minkowski: $\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} \leq \left(\sum |a_k|^p \right)^{1/p} + \left(\sum |b_k|^p \right)^{1/p}$

Hölder

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Panel 3

You define: (X, ρ)

ℓ_p

$C^{p}[a,b]$ $f: [a,b] \to \mathbb{R}$ cont.,
$\rho(f,g) = \left(\int |f-g|^p \, dt\right)^{1/p}$

$\left(C[a,b], \rho\right)$

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Panel 4

$X = C[0,2]$, $\rho(x,y) = \max_{0 \le t \le 2} \left(|x(t) - y(t)| \right)$.

Define $f: C[0,2] \to C[0,2]$, $f(x) = \int_0^t x(s)\,ds$

a) f onto?

every one in Range has a friend in Domain No!

$\cos(t)$ has no pre image, because $f(x)(0) = \int_0^0 x(s)\,ds = 0$
$\Rightarrow f(x) \ne \cos(t)$

b) f one-one?

$f(x) = f(y) \Rightarrow \int_0^t x(s)\,ds = \int_0^t y(s)\,ds \Rightarrow \int_0^t x(s) - y(s)\,ds = 0 \; \forall t$

$\frac{d}{dt}: \; x(t) - y(t) = 0 \quad \forall t$

$\Rightarrow$ yes!

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Panel 5

c) Is f continuous? Given $\varepsilon > 0$ pick $\delta = \varepsilon/2$

s.t. if $\rho(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$ goal!

Want: $\rho(f(x), f(y)) = \max_{0 \le t \le 2} \left| \int_0^t x(s)\,ds - \int_0^t y(s)\,ds \right| < \varepsilon$ goal

$\Rightarrow \max_{0 \le t \le 2} \left| \int_0^t x(s) - y(s)\,ds \right| < \varepsilon$

$$\left| \int_0^t x(s) - y(s)\,ds \right| \le \int_0^t |x(s) - y(s)|\,ds \le \int_0^t \max |x(s) - y(s)|\,ds$$

$$\le \int_0^t \delta\,ds \le \int_0^2 \delta\,ds = 2\delta \quad \forall t$$

Pick $\delta = \varepsilon/2$

Panel 6

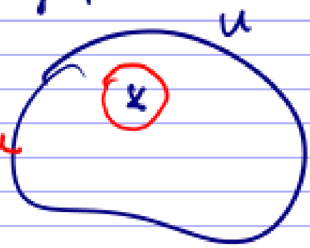
Using our metric we can define open/closed sets:

Def: Let (X, ρ) be a metric space. The open sphere centered at x_0 with radius r is $\rho(x_0, x) < r$

The set $O_\varepsilon(x_0) = \{\rho(x_0, x) < \varepsilon\}$ is an ε-nbhd of x_0.

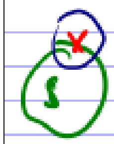
A set $U \subset X$ is open if every point $x \in U$ has a nbhd $O_\varepsilon(x) \subset U$

A set $C \subset X$ is closed if C^c is open



Panel 7

Def: A point $x \in X$ is a boundary point of $S \subset X$ if every nbhd. $O_\varepsilon(x)$ contains points in S and in S^c



A point $x \in X$ is a limit point of $S \subset X$ if every nbhd. $O_\varepsilon(x)$ contains inf. many points of S.

The closure of a set S is $S \cup \{\text{boundary points}\}$ called $\bar{S}$

Ex: $\mathbb{Q} \subset \mathbb{R}$: boundary: $\mathbb{R}$ limit points $\mathbb{R}$

$\bar{\mathbb{Q}} = \mathbb{R}$

Find $S \subset \mathbb{R}$ with a boundary point that's no limit pt.

$[0,1] \cup \{2\}$

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Panel 8

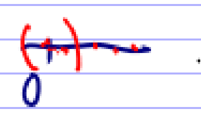
Convergence and Limits

Def: A sequence $\{x_n\}$ in a metric space converges to $x \in X$ if for every open nbhd. $O_\varepsilon(x)$ there exist N s.t. $x_k \in O_\varepsilon(x)$ $\forall k > N$

Thm: (i) $x_n \to x$ iff $\rho(x_n, x) \to 0$ as $n \to \infty$

(ii) Limit is unique

(iii) Every subsequence conv. to x also. Ex $\frac{1}{n} \to 0$



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Panel 9

Dense and Separable Sets

Def: Take $A, B \subset X$. Then A is dense in B if $B \subset \bar{A}$

A set A is (everywhere) dense if $\bar{A} = X$

A set A is nowhere dense if it is dense in no open sphere.

Def: A metric space (X, ρ) is separable if it contains a countable dense subset!

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Panel 10

Show that $\mathbb{R}, \mathbb{R}^n_0, \mathbb{R}^n_1, \mathbb{R}^n_2, \ell_2, C[a,b], C^2[a,b]$ are all separable.

(1) $\mathbb{Q}$ is dense in $\mathbb{R}$ ✓ (?)

(2) $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ in any metric

(3) $(x_1, x_2, x_3, \dots)$, $x_j \in \mathbb{Q}$, but not countable!
$(x_1, x_2, x_3, \dots x_n, 0, 0 \dots)$, $x_j \in \mathbb{Q}$, i.e. all finite sequences of $\mathbb{Q}$, filled in with 0's.
$E_j = (x_1, x_2, \dots, x_j, 0, 0, 0)$, $x_k \in \mathbb{Q}$. Each E_j is countable (finite cross-product of countable sets). $\bigcup_{j=1}^{\infty} E_j$ is countable

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Panel 11

Take $S = \{$all finite sequences of rational #'s $\}$.
Take any $x \in \ell_2$. Given any $\varepsilon > 0$ need to find
$s \in S$ s.t. $\rho(s, x) < \varepsilon$

Want: $\sum_{i=1}^{\infty} (s_i - x_i)^2 < \varepsilon$ ∴ Hint

Know: $\sum_{j=1}^{\infty} x_j^2$ conv. $\to \sum_{j=N}^{\infty} x_j^2 < \frac{\varepsilon}{2}$

$$\underbrace{(s_1 - x_1)^2}_{\varepsilon/2} + \underbrace{(s_2 - x_2)^2}_{\varepsilon/4} + \dots + \underbrace{(s_N - x_N)^2}_{\varepsilon/2^N} + \underbrace{(0 - x_{N+1})^2 + \dots}_{\varepsilon/2}$$

$$\leq \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} + \frac{\varepsilon}{2}$$

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

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Panel 12

$C[a,b]$ is separable:

$\mathcal{P} = \{$polyn. with rational coefficients$\}$

Thm: Every cont. function on $[a,b]$ can be approximated by polynomials. (no proof given - YET)

Every $p_n(x) = a_n x^n + \dots + a_0$ can be approx. by polyn. with rational coefficients.

$\mathcal{P}$ is countable. Why?

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Panel 13

Ex: Consider the set m of all bounded sequences with $\rho(x,y) = \sup(|x_k - y_k|)$. Then m is a metric space. Is it separable?

$\{1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \ldots\} = \{1/\sqrt{n}\} \in m$. Is it in l_2?

$l_2 \subset m$, but $m \cancel{\subset} l_2$ m is NOT separable:

$\{$take all sequences of 0's and 1's only$\} = S$.

$S \subset m$ ✓ $\rho(x,y) = 1$ $\forall x,y \in S$

Rest next time!

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