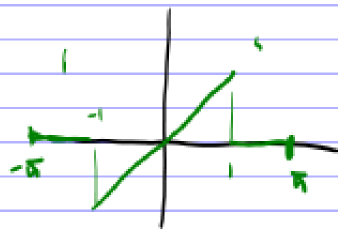
Panel 1

$$f(x) = \begin{cases} x & \text{if } x \in [-1,1] \\ 0 & \text{else} \end{cases} \quad \text{on } [-\pi,\pi]$$

① Does f have Fourier Series?



$= 0$

f is (piecewise) continuous

all derivatives are cont.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{-\pi}^{-1} \overset{0}{\cancel{f(x)}}\,dx + \int_{-1}^{1} \cancel{f(x)}\,dx + \int_{1}^{\pi} \overset{0}{\cancel{f(x)}}\,dx\right) = \frac{1}{2\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-1}^{1} x\cos(nx)\,dx = 0 \quad \text{(odd function)}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-1}^{1} x\sin(nx)\,dX =$$

1

Panel 2

$$\frac{1}{\pi}\int_{-1}^{1} x\sin(nx)\,dx = \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx)\Big|_{-1}^{1} + \frac{1}{n}\int_{-1}^{1}\cos(nx)\,dx\right] =$$

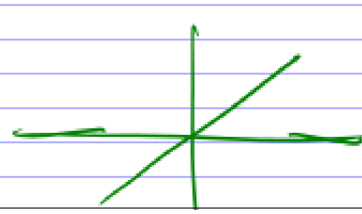
$u = x \qquad u' = 1$

$v' = \sin(nx) \qquad v = -\frac{1}{n}\cos(nx)$

$$= \frac{1}{\pi}\left[-\frac{1}{n}\cos(n) - \frac{1}{n}\cos(-n) + \frac{1}{n}\,\frac{1}{n}\,\sin(nx)\Big|_{-1}^{1}\right] =$$

$$= \frac{1}{\pi}\left(-\frac{2}{n}\cos(n) + \frac{1}{n^2}\left(\sin(n) - \sin(-n)\right)\right) = \frac{1}{\pi}\left(-\frac{2}{n}\cos(n) + \frac{2}{n^2}\sin(n)\right)$$

$$a_n = \frac{1}{\pi}\left(\frac{2}{n^2}\sin(n) - \frac{2}{n}\cos(n)\right)$$



$$g(x) = \sum a_n \sin(nx)$$

2

Panel 3

Ex: Set of continuous functions on $[a,b]$
with $\rho(f,g) = \left(\int (f(t)-g(t))^2 dt\right)^{1/2}$

Denoted by $C^2[a,b]$

Schwarz: $\int f(t)g(t)\,dt \le \left(\int f^2 dt\right)\left(\int g^2 dt\right)$

→ triangle inequality!

Each "point" in metric space is a function!

3

Private Freehand 3

3

Panel 4

Ex: Set of all inf. sequences $x = (x_1, x_2, \dots)$, $x_j \in \mathbb{R}$, s.t.

$$\sum_{k=1}^{\infty} x_k^2 < \infty$$

is a metric space called ℓ^2

$$\rho(x,y) = \left(\sum_{k=1}^{\infty} (x_k - y_k)^2\right)^{1/2}$$

Requires: if $\sum_{k=1}^{\infty} x_k^2 < \infty$ and $\sum y_k^2 < \infty$

$\Rightarrow \sum (x_k - y_k)^2 < \infty$ ✓ true by comp. test.

$$(x_k \pm y_k)^2 = x_k^2 \pm 2x_k y_k + y_k^2 \leq 2(x_k^2 + y_k^2)$$

because $0 \leq x_k^2 \mp 2x_k y_k + y_k^2 = (x_k \mp y_k)^2$

4

Panel 5

Triangle Inequality: $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$

$$\left(\sum (x_k - z_k)^2\right)^{1/2} \leq \left(\sum (x_k - y_k)^2\right)^{1/2} + \left(\sum (y_k - z_k)^2\right)^{1/2}$$

follows from case $\mathbb{R}^n_2$, i.e.

all $(x_1, x_2, \dots, x_n)$, $\rho(x,y) = \left(\sum_{j=1}^{n} (x_j - y_j)^2\right)^{1/2}$

by $n \to \infty$

5

Panel 6

Ex: $\mathbb{R}^n_p$, $\rho(x,y) = \left(\sum_{k=1}^{n} (x_k - y_k)^p\right)^{1/p}$, $p \geq 1$

$\mathbb{R}^n_0$

$\mathbb{R}^n_1$ $\qquad \mathbb{R}^n_p$

$\mathbb{R}^n_2$

Minkowski's Inequality

Triangle: $\left(\sum_{k=1}^{n} |x_k - z_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{1/p} + \left(\sum_{k=1}^{n} |y_k - z_k|^p\right)^{1/p}$

Depends on $\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \cdot \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q}$,

$1/p + 1/q = 1$ Hölder's Inequality

Note: Hölder with $p=2 \Leftrightarrow$ Cauchy-Schwarz!

6

Panel 7

Summary:

$\mathbb{R}^n_0$ what is a "point" in each,

$\mathbb{R}^n_1$ what is $\rho(x,y)$ - metric

$\mathbb{R}^n_2$

$C[a,b]$

$C^2[a,b]$

ℓ_2

$\mathbb{R}^n_p$

7

Panel 8

Cauchy - Schwarz

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_1^n a_k^2\right)\left(\sum_1^n b_k^2\right)$$

Schwarz $\left(\int f(t)g(t)\,dt\right)^2 \leq \left(\int f^2\,dt\right)\left(\int g^2\,dt\right)$

Minkowski $\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_1^n |a_k|^p\right)^{1/p} + \left(\sum_1^n |b_k|^p\right)^{1/p}$

Hölder $\sum |a_k b_k| \leq \left(\sum |a_k|^p\right)^{1/p} \left(\sum |b_k|^q\right)^{1/q}, \quad \frac{1}{p} + \frac{1}{q} = 1$

$p, q \geq 1$

8

Panel 9

<u>You define:</u>

ℓ_p

$C^p[a,b]$

9

Panel 10

Now we can define familiar concepts in new settings:

Def: $f: X \to Y$, $X = (X, \rho)$ and $Y = (Y, \rho')$ metric spaces. Then f is continuous at $x_0 \in X$ if:

given $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$\rho'(f(x), f(x_0)) < \varepsilon$ whenever $\rho(x, x_0) < \delta$

Note: $f: X \to \mathbb{R}$ is a function.

$f: X \to Y$ is a functional

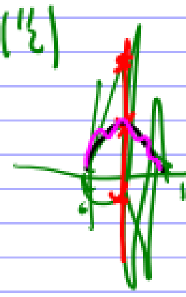
10

Panel 11

Ex: $f: C^0[0,1] \to \mathbb{R}$, $f(g(x)) = g(\tfrac{1}{2})$.

Continuous? Yes

$f(x^2) = \tfrac{1}{4}$, $f(x^3) = \tfrac{1}{8}$, $f(\sin(x)) = \sin(\tfrac{1}{2})$



Is it cont. at $x_0 \in C[0,1]$

if $\rho(x_0, g) < \delta \Rightarrow \rho(f(x_0), f(g)) < \varepsilon$

if $\max(|x_0(t) - g(t)|) < \delta \Rightarrow |x_0(\tfrac{1}{2}) - g(\tfrac{1}{2})| < \varepsilon$

Take any $\varepsilon > 0$, pick $\delta = \varepsilon$.

$\Rightarrow |x_0(\tfrac{1}{2}) - g(\tfrac{1}{2})| \le \max_{t \in [0,1]} |x_0(t) - g(t)| < \delta = \varepsilon$

11

Panel 12

Ex: $f: l_2 \to l_2$, $f((x_1, x_2, \dots)) = (x_2, x_3, x_4, \dots)$

Ex: $f((1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$ shift operator

$\sum (\frac{1}{n})^2 < \infty$

f cont? at $x_0 = (x_1^0, x_2^0, x_3^0, \dots)$ (Pup)

if $\rho(x_0, x) < \delta \overset{\text{want}}{\Rightarrow} \rho(f(x_0), f(x)) < \varepsilon$

$\left(\sum_{j=1}^{\infty} (x_j^0 - x_j)^2\right)^{1/2} < \delta \Rightarrow \left(\sum_{j=2}^{\infty} (x_j^0 - x_j)^2\right)^{1/2} < \varepsilon$

domain

pick $\delta = \varepsilon$.

12

Private Freehand 12

12

Panel 13

Ex: $L : C[0,2] \to C[0,2]$, $L(f) = \int_0^x f(t)\,dt$

continuous?

13