

Panel 1

Ex: Set of continuous functions on $[a, b]$
 with $\rho(f, g) = \left(\int_a^b (f-g)^2 dx \right)^{1/2}$
 Denote as $C^2[a, b]$
Triangle Ineq: $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$
 Schwarz: $\left(\int_a^b fg dx \right)^2 \leq \left(\int_a^b f^2 dx \right) \cdot \left(\int_a^b g^2 dx \right)$

Panel 2

$\int_a^b \int_a^b (f(s)g(t) - h(s)g(t))^2 ds dt =$
 $\int_a^b \int_a^b f^2(s)g^2(t) ds dt - 2 \int_a^b \int_a^b f(s)g(t)h(s)g(t) ds dt + \int_a^b \int_a^b h^2(s)g^2(t) ds dt$
 $\int_a^b g^2(t) \cdot \int_a^b f^2(s) ds dt - 2 \int_a^b g(t)h(t) \left(\int_a^b f(s)g(s) ds \right) dt +$
 $\int_a^b h^2(t) \int_a^b g^2(s) ds dt - 2 \int_a^b g(t)h(t) \left(\int_a^b f(s)h(s) ds \right) dt + \int_a^b h^2(t) \int_a^b g^2(s) ds dt$
 $2 \int_a^b g^2(t) dx \cdot \int_a^b f^2(t) dx - 2 \left(\int_a^b f(t)h(t) dx \right)^2$
 $\Rightarrow 2 \left(\int_a^b g^2(t)h^2(t) dx \right)^2 = 2 \left(\int_a^b g^2(t) dx \right) \left(\int_a^b h^2(t) dx \right) - 2 \int_a^b (f-h)^2 dx \leq$

Panel 3

$\left(\int_a^b f(t)g(t) dx \right)^2 = \dots \leq \left(\int_a^b f^2 dx \right) \left(\int_a^b g^2 dx \right)$
 Define $h(t) = \int_a^b f^2 dx + t^2 - 2 \int_a^b f(t)g(t) dx + \int_a^b g^2 dx$
 $\cdot \left(\int_a^b f^2 dx + \int_a^b g^2 dx \right)^2 \geq 0 \quad \forall t$
 Discrim. ≤ 0
 $4 \left(\int_a^b f(t)g(t) dx \right)^2 - 4 \left(\int_a^b f^2 dx \right) \left(\int_a^b g^2 dx \right) \leq 0$
 $\Rightarrow \left(\int_a^b fg dx \right)^2 \leq \left(\int_a^b f^2 dx \right) \left(\int_a^b g^2 dx \right)$

Panel 4

Know: $\left(\int_a^b fg dx \right)^2 \leq \left(\int_a^b f^2 dx \right) \left(\int_a^b g^2 dx \right) \checkmark$
 want: $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$
 $\left(\int_a^b (f-h)^2 dx \right)^{1/2} \leq \left(\int_a^b (f-g)^2 dx \right)^{1/2} + \left(\int_a^b (g-h)^2 dx \right)^{1/2}$
 $\left(\int_a^b (a+b)^2 dx \right)^{1/2} \leq \left(\int_a^b a^2 dx \right)^{1/2} + \left(\int_a^b b^2 dx \right)^{1/2}$
 $\int_a^b (a+b)^2 dx = \int_a^b a^2 dx + 2 \int_a^b ab dx + \int_a^b b^2 dx$
 $\leq \int_a^b a^2 dx + 2 \left(\int_a^b a^2 dx \right)^{1/2} \left(\int_a^b b^2 dx \right)^{1/2} + \int_a^b b^2 dx$

Panel 5

$$\begin{aligned}\int (a+s)^2 dx &= \int a^2 dx + 2 \int a s dx + \int s^2 dx \\ &\leq \int a^2 dx + 2 \left(\int a^2 dx \right)^{1/2} \left(\int s^2 dx \right)^{1/2} + \int s^2 dx \\ &= \left[\left(\int a^2 dx \right)^{1/2} + \left(\int s^2 dx \right)^{1/2} \right]^2 \\ \left(\int (a+s)^2 dx \right)^{1/2} &\leq \left(\int a^2 dx \right)^{1/2} + \left(\int s^2 dx \right)^{1/2}\end{aligned}$$

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