Panel 1

Next Quiz: Fourier Series + Convergence Types

Def: Mean-Square Conv., conv. a.e. ✓

Fourier Series for $f: [-\pi,\pi] \to \mathbb{R}$

Thm: When does f_n conv. to f, f_n are Fourier polyn.

i.e. for what f is $f(x) = a_0 + \sum a_n \cos(nx) + b_n \sin(nx)$

Ex: $f(x) =$ ____. Find its Fourier Series

Here's f_n. Does it conv. a.e. / mean squared / uniformly to f.

1

Panel 2

Metric Spaces

Def: A metric space is a pair (X, ρ)

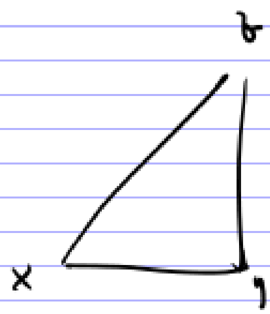
X is a set, ρ is function on $X \times X$ s.t.

(1) $\rho(x,y) \geq 0$

(2) $\rho(x,y) = 0 \Rightarrow x = y$

(3) $\rho(x,y) = \rho(y,x)$

(4) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$



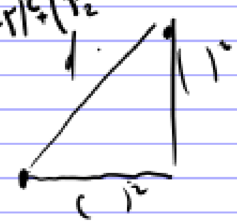
2

Panel 3

Ex: $(\mathbb{R},\rho)$, $\rho(x,y)=|x-y|$ is old-fashioned * line.

$(\mathbb{R}^n,\rho)$, $\rho(x,y)=\left(\sum_{j=1}^{n}(x_j-y_j)^2\right)^{1/2}$

e.g. $\mathbb{R}^2$: $\rho((5,6),(1,3))=\sqrt{(1-5)^2+(3-6)^2}$



space we live in $(n=3)$

$(\mathbb{R}^n,\rho)$, $\rho(x,y)=\sum_{j=1}^{n}|x_j-y_j|$

$(n=2)$: $\rho((x_1,x_2),(y_1,y_2))=|x_1-y_1|+|x_2-y_2|$ is metric

3

Panel 4

To get a feel for (X,ρ), draw $\rho(x,0)\le 1$ "unit ball"

$(\mathbb{R}^2,\rho)$, $\rho(x,y)=\left(\sum(x_j-y_j)^2\right)^{1/2}$

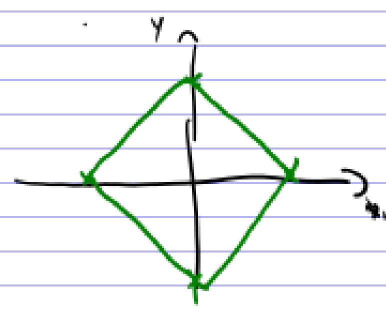
(x_1,x_2) s.t. $\rho(x,0)=\sqrt{(x_1-0)^2+(x_2-0)^2}\le 1$

$\sqrt{x_1^2+x_2^2}\le 1$

$(\mathbb{R}^2,\rho)$: $\rho(x,y)=\sum|x_j-y_j|$

(x_1,x_2): $\rho(x,0)=|x_1|+|x_2|\overset{=1}{\le} 1$

$|x|+|y|=1$

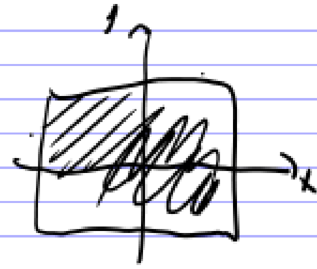


4

Panel 5

$(\mathbb{R}^n, \rho)$: $\rho(x,y) = \max_{1 \le j \le n}(|x_j - y_j|)$

$\mathbb{R}^2$: $\rho(x,0) \le 1 \Leftrightarrow \max(|x_1|, |x_2|) \le 1$



$\mathbb{R}^n_2$ $\sqrt{(\,|\,|^2 + (\,)^2)}$

$\mathbb{R}^n_1$ $|\quad|$

$\mathbb{R}^n_\infty = \max$

5

Panel 6

Is $\rho(x,y) = \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2}$ a metric

Hard: triangle:

$$\sqrt{\sum_{j=1}^{n}(x_j - z_j)^2} \le \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2} + \sqrt{\sum_{j=1}^{n}(y_j - z_j)^2}$$

$$\Leftrightarrow \sqrt{\sum (a_j + b_j)^2} \le \sqrt{\sum a_j^2} + \sqrt{\sum_{j=1}^{n} b_j^2}$$

let $x_j - y_j = a_j$, $y_j - z_j = b_j$

Cauchy-Schwarz $\left(\sum_{j=1}^{n} a_j b_j\right)^2 \le \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$

6

Panel 7

$$\left(\sum_{j=1}^{n} a_j s_j\right)^2 \le \left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} s_j^2\right)$$

Start with $\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{n}(a_j s_i - a_i s_j)^2 =$

(e.g. $n=2$. $\underbrace{(a_1 s_1 - a_1 s_1)^2}_{0} + (a_1 s_2 - a_2 s_1)^2 + (a_2 s_1 - a_1 s_2)^2 + \underbrace{(a_2 s_2 - a_2 s_2)^2}_{0}$

$2(a_1 s_2 - a_2 s_1)^2$)

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\left(a_j^2 s_i^2 - 2 a_j s_i a_i s_j + a_i^2 s_j^2\right) =$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_j^2 s_i^2 - 2\sum_{j=1}^{n}\sum_{i=1}^{n} a_j s_i a_i s_j + \sum_{j=1}^{n}\sum_{i=1}^{n} a_i^2 s_j^2$$

7

Panel 8

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_j^2 s_i^2 - 2\sum_{j=1}^{n}\sum_{i=1}^{n} a_j s_i a_i s_j + \sum_{j=1}^{n}\sum_{i=1}^{n} a_i^2 s_j^2 =$$

$a_1^2 s_1^2 + a_1^2 s_2^2 + \dots a_1^2 s_n^2 + \underline{a_2^2} s_1^2 + \underline{a_2^2} s_2^2 + \dots + \underline{a_2^2} s_n^2 + \dots$

$\underline{a_1^2 (s_1^2 + s_2^2 + \dots s_n^2)} + a_2^2 \underline{(s_1^2 + s_2^2 + \dots + s_n^2)} + \dots$

$(s_1^2 + s_2^2 + \dots + s_n^2)(a_1^2 + a_2^2 + \dots + a_n^2)$

$$\sum_{j=1}^{n} a_j^2 \sum_{i=1}^{n} s_i^2 - 2\sum_{j=1}^{n} a_j s_j \sum_{i=1}^{n} a_i s_i + \sum_{j=1}^{n} s_j^2 \sum_{i=1}^{n} a_i^2 =$$

$$\left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{i=1}^{n} s_i^2\right) - 2\left(\sum_{j=1}^{n} a_j s_j\right)\left(\sum_{i=1}^{n} a_i s_i\right) + \left(\sum_{j=1}^{n} s_j^2\right)\left(\sum_{j=1}^{n} a_j^2\right).$$

8

Panel 9

$$\left(\sum_{j=1}^n a_j^2\right)\left(\sum_{i=1}^n s_i^2\right) - 2\left(\sum_{j=1}^n a_j s_j\right)\left(\sum_{i=1}^n a_i s_i\right) + \left(\sum_{j=1}^n s_j^2\right)\left(\sum_{j=1}^n a_j^2\right) =$$

$$\left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n s_k^2\right) - 2\left(\sum_{k=1}^n a_k s_k\right)\left(\sum_{k=1}^n a_k s_k\right) + \left(\sum_{k=1}^n s_k^2\right)\left(\sum_{k=1}^n a_k^2\right) =$$

$$\cancel{2}\left(\sum_{k=1}^n a_k^2\right)\left(\sum_{k=1}^n s_k^2\right) - \cancel{2}\left(\sum_{k=1}^n a_k s_k\right)^2 = \frac{1}{2}\sum\sum (a_j s_i - a_i s_j)^2$$

$$\left(\sum a_k^2\right)\left(\sum s_k^2\right) - \text{pos} \le \qquad \left(\sum_{k=1}^n a_k s_k\right)^2 \le \left(\sum a_k^2\right)\left(\sum s_k^2\right)$$

Cauchy Schwarz!

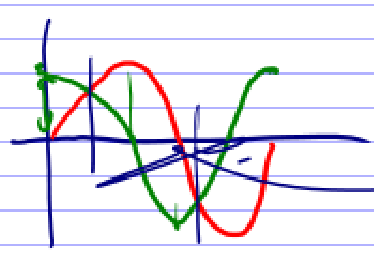
9

Panel 10

$C[0,2\pi]$, $\quad \rho(f,g) = \max_{0\le t\le 2\pi} |f(t)-g(t)|$

Ex: $\rho(\cos(t), \sin(t)) = \max|\cos(t) - \sin(t)| \geq 1$

$|\cos(t)-\sin(t)|$, $t \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

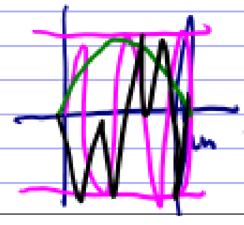


$d(t) = \sin(t) - \cos(t)$,

~~$d'(t) = \cos(t)+\sin(t) = 0$~~, $t=$

$d\left(\frac{3\pi}{4}\right) = \sqrt{2}$

$\rho(f,0)\le 1$, $\quad f:[0,2\pi]\to \mathbb{R}$, $\max_{0\le t\le 2\pi}|f(t)| \le 1$



10

Panel 11

Ex: Set of continuous functions on $[a,b]$

with $\rho(f,g) = \left[\int_a^b (f(t)-g(t))^2 dt \right]^{1/2}$

Call this $C^2[a,b]$

Triangle Ineq. depends Schwarz Inequ:

$$\left(\int_a^b f(t) g(t) dt \right)^2 \leq \left(\int_a^b f(t)^2 dt \right) \left(\int_a^b g(t)^2 dt \right)$$

Start with $\int_a^b \int_a^b (f(t)g(s) - f(s)g(t))^2 ds\, dt$

11