Panel 1

Next Quiz: Fourier Series + Convergence Types

Def. of Fourier Series

When is F.S. conv. to f

Compute the F.S. of $f(x) = \begin{cases} - \\ - \end{cases}$

conv. a.e.
mean-square conv.
pointwise conv.
unif. conv.

Does f_n conv. to f.

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Panel 2

Metric Spaces

(see KF p 37 ff)

Def. A metric space is a pair (X, ρ), X a set an ρ a function defined on $X \times X$:

$\rho(x,y) \geq 0$

$\rho(x,y) = 0 \Rightarrow x = y$

$\rho(x,y) = \rho(y,x)$

$\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ — triangle inequality

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Panel 3

Ex: $(\mathbb{R}^n, \rho)$, $\rho(x,y) = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$ ✓

$\mathbb{R}^n_2$ (usual Euclidean space) → Prove triangle inequality

Ex: $(\mathbb{R}^n, \rho)$, $\rho(x,y) = \sum_{k=1}^{n} |x_k - y_k|$

$\mathbb{R}^n_1$

Ex: $(\mathbb{R}^n, \rho)$, $\rho(x,y) = \max_{1 \le k \le n} \{ |x_k - y_k| \}$

$\mathbb{R}^n_0$

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Panel 4

Lemma: (Cauchy-Schwarz Inequality)

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 \le \left(\sum_{k=1}^{n} a_k^2 \right) \cdot \left(\sum_{k=1}^{n} b_k^2 \right) ✓$$

Hint: $$\left(\sum_{k=1}^{n} a_k b_k \right)^2 = \left(\sum_1^n a_k^2 \right)\left(\sum_1^n b_k^2 \right) - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2 ✓$$
$$\le \left(\sum a_k^2 \right)\left(\sum b_k \right)^2$$ because of positive

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_i b_j - a_j b_i)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^2 b_j^2 - 2 a_i b_j a_j b_i + a_j^2 b_i^2 =$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^2 b_j^2 - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j b_i b_j + \underline{\sum_{i=1}^{n} \sum_{j=1}^{n} a_j^2 b_i^2}$$

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Panel 5

$i=1,\ j=1 \qquad i=1, j=2$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 = a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + \ldots a_1^2 b_n^2 +$$
$$a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + \ldots a_2^2 b_n^2 +$$
$$\vdots$$
$$= a_1^2\underline{(b_1^2 + b_2^2 + \ldots + b_n^2)} + a_2^2\underline{(b_1^2 + \ldots + b_n^2)} + \ldots$$
$$= (b_1^2 + b_2^2 + \ldots + b_n^2)(a_1^2 + a_2^2 + \ldots + a_n^2)$$
$$= \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_j^2 b_i^2 = \left(\sum_{k=1}^{n} b_k^2\right)\left(\sum_{k=1}^{n} a_k^2\right)$$

(B) $$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j b_i b_j = \sum_{i=1}^{n} a_i b_i \sum_{j=1}^{n} a_j b_j = \left(\sum_{k=1}^{n} a_k b_k\right)\cdot\left(\sum_{k=1}^{n} a_k b_k\right) = \left(\sum_{k=1}^{n} a_k b_k\right)^2$$

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Panel 6

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (a_i b_j - a_j b_i)^2 = 2\sum_{k=1}^{n} a_k^2 b_k^2 - 2\left(\sum_{k=1}^{n} a_k b_k\right)^2$$

$$\Rightarrow \left(\sum a_k b_k\right)^2 = \sum a_k^2 b_k^2 - \frac{1}{2}\sum\sum(\qquad)^2 \quad \underline{\text{qed}}$$

Much better proof: Define $f(x)$ as

$$f(x) = \left(\sum_{k=1}^{n} a_k^2\right)x^2 - 2\left(\sum_{k=1}^{n} a_k b_k\right)x + \sum_{1}^{n} b_k^2 =$$
$$= \sum_{k=1}^{n}(a_k x - b_k)^2 \geq 0$$

$$f(x) = Ax^2 + Bx + C \geq 0$$

$$\Rightarrow x_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$4\left(\sum a_k b_k\right)^2 - 4\left(\sum a_k^2\right)\left(\sum b_k^2\right) \leq 0$$

$$\Rightarrow B^2 - 4AC \leq 0$$

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Panel 7

Ex: Draw "unit ball" in $\mathbb{R}^2_2$, $\mathbb{R}^2_1$, $\mathbb{R}^2_\infty$

$\mathbb{R}^2_2$: $\rho(x,y) = \sqrt{\sum_{k=1}^{2} (x_k-y_k)^2}$ $\qquad \rho(x,0)\le 1$

$$\sqrt{(x_1-0)^2+(x_2-0)^2} \le 1$$

$$x_1^2+x_2^2 \le 1$$

$\mathbb{R}^2_1$: $\rho(x,y) = \sum_{i=1}^{2} |x_i-y_i|$ $\qquad \rho(x,0)$

$|x_1-0|+|x_2-0| \le 1 \Rightarrow |x_1|+|x_2| \le 1$ Graph (diam)

$\mathbb{R}^2_\infty$: $\rho(x,y) = \max|x_i-y_i| \Rightarrow \max(|x_1|,|x_2|) \le 1$

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Panel 8

Ex: Set of cont. function on $[a,b]$ with

$$\rho(f,g) = \max_{a\le t\le b}\{|f(t)-g(t)|\}$$

is metric space. Prove it.

a) $\rho(f,g)\ge 0$

b) $\rho(f,g)=0 \Rightarrow \max|f(t)-g(t)|=0 \Rightarrow f(t)-g(t)\equiv 0$

c) $\rho(f,g)=\rho(g,f)$

d) $\rho(f,h)\le \rho(f,g)+\rho(g,h)$

$|f(t)-h(t)| \le |f(t)-g(t)|+|g(t)-h(t)| \Rightarrow$ true for max!

$|f-g+g-h| \le |f-g|+|g-h| \quad \forall t$

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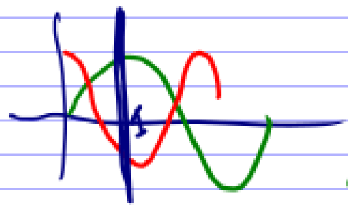
Panel 9

$C^0[0,1]$

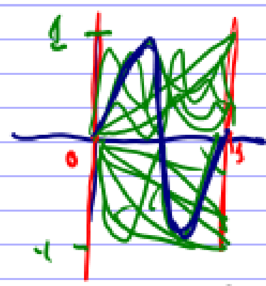
$\rho(1,0)$ for $1 = x^2$, $C[0,1]$.

$\rho(f,0) = \max\limits_{0 \le t \le 1} \{ |f(t) - 0| \} = \max\limits_{t \in [0,1]} \{ |t^2| \} = 1$

$\rho(\sin(t), \cos(t)) = \max \{ |\sin(t) - \cos(t)| \} = 1$



$\rho(f,0) < 1$, $f: [0,1] \to \mathbb{R}$



$\rho(f,0) = \{ f: [0,1] \to [-1,1] \}$

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Panel 10

HW: What is max. distance between $\sin(x), \cos(x)$, $x \in [0, 1]$

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Panel 11

Ex: Set of continuous functions on $[a,b]$ with $\rho(f,g) = \left(\int_a^b (f-g)^2 dt \right)^{1/2}$

Denote by $C^2[a,b]$. Is this a metric?

Proof depends on Schwarz ineqn:

$$\left[\int_a^b fg \, dx \right]^2 \le \left(\int_a^b f^2 dx \right) \left(\int_a^b g^2 dx \right)$$

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