Panel 1

Last Time: f cont. on $[a,b]$, f' piecewise diff'ble with f' cont. on each piece.

$f_n \rightrightarrows f$ uniformly, f_n = Fourier Series

$f: [-\pi, \pi] \to \mathbb{R}$

$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$

$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx$

$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx$

$f: [-L, L] \to \mathbb{R}$

$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi}{L} x)\,dx$

$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi}{L} x)\,dx$

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Panel 2

New types of Convergence:

$f_n \to f$ converges pointwise

$f_n \to f$ a.e. converges ptwise except of a set of measure zero

$f_n \xrightarrow{L_2} f$ $\int_a^b |f_n(x) - f(x)|^2\,dx \to 0$ mean square

$f_n \rightrightarrows f$ converges uniformly

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Panel 3

$f_n(x) = 0 \quad \forall n$

$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{else} \end{cases}$

$\int_0^1 |f_n(x) - f(x)|^2 dx \to 0$

$\int_0^1 f(x)\, dx$ ☹ d.n.e. as R-int.

$$\int_0^1 |f_n(x) - f(x)|^2 d\mu = \int_0^1 f(x)\, d\mu = \int_{\mathbb{Q}\cap[0,1]} f(x)\, d\mu + \int_{\mathbb{I}\cap[0,1]} f(x)\, d\mu$$

$$= \underbrace{0}_{\mu(\mathbb{Q}\cap[0,1])=0} + \underbrace{0}_{f=0}$$

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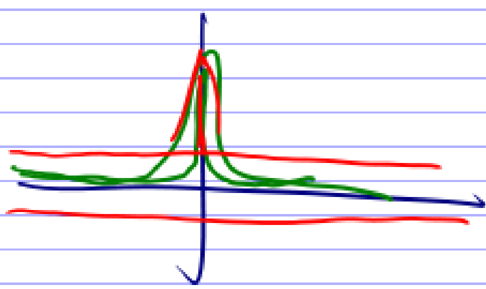
Panel 4



not uniform conv.

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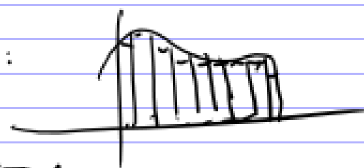
Panel 5

How to compute Fourier Series from 2^n-samples

1. Sample f at 2^n points $[0, 2\pi]$
2. Store f's in a list A (array)
   Setup array B of size 2^n containing 0's
3. $(A, B) \longrightarrow \boxed{\text{FFT}} \longrightarrow (A, B)$ sin-coeff, cos-coeff

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$ could :



Fast Fourier Transform FFT

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Panel 6

```java
public static void computeFFT(int sign, int n, double ar[], double ai[])
{
    double scale = 2.0 / (double) n;
    int i, j;
    for (i = j = 0; i < n; ++i)
    {
        if (j >= i)
        {
            double tempr = ar[j] * scale;
            double tempi = ai[j] * scale;
            ar[j] = ar[i] * scale;
            ai[j] = ai[i] * scale;
            ar[i] = tempr;
            ai[i] = tempi;
        }
        int m = n / 2;
        while ((m >= 1) && (j >= m))
        {
            j -= m;
            m /= 2;
        }
        j += m;
    }

    int mmax, istep;
    for (mmax = 1, istep = 2 * mmax; mmax < n; mmax = istep, istep = 2 * mmax)
    {
        double delta = sign * Math.PI / (double) mmax;
        for (int m = 0; m < mmax; ++m)
        {
            double w = m * delta;
            double wr = Math.cos(w);
            double wi = Math.sin(w);
            for (i = m; i < n; i += istep)
            {
                j = i + mmax;
                double tr = wr * ar[j] - wi * ai[j];
                double ti = wr * ai[j] + wi * ar[j];
                ar[j] = ar[i] - tr;
                ai[j] = ai[i] - ti;
                ar[i] += tr;
                ai[i] += ti;
            }
        }
        mmax = istep;
    }
}
```

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Panel 7

```java
import bgw.math.FFT;

public class FFTExample
{
    public static double min = 0.0;
    public static double max = 2*Math.PI;
    public static int N = 1024;

    public static double f(double x)
    {
        return (x - Math.PI)*(x-Math.PI);
    }

    public static void main(String args[])
    {
        double[] A = new double[N];
        double[] B = new double[N];

        evaluateFunction(A, B);
        FFT.computeFFT(A, B);
        displayCoefficients(A, B, 10);
    }

    public static void evaluateFunction(double[] A, double[] B)
    {
        double x = min;
        double dx = (max - min) / N;
        for (int i = 0; i < N; i++)
        {
            A[i] = f(x);
            B[i] = 0.0;
            x += dx;

        }
    }

    public static void displayCoefficients(double[] A, double B[], int maxN)
    {
        System.out.println("a[0] = " + (A[0]/2));
        for (int i = 1; i < maxN; i++)
        {
            System.out.println("a[" + i + "] = " + A[i]);
            System.out.println("b[" + i + "] = " + B[i]);
        }
    }
}
```

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Panel 8

Ex: $f(x) = (x-\pi)^2$ on $[0, 2\pi]$

Problem: $[0, 2\pi)$ not $[-\pi, \pi)$. Translate by π:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\pi)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)^2\,dx = 3.289$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\pi)\cos(x)\,dx = -4$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\pi)\sin(nx)\,dx = 0$$

$a_2 = 1$

$a_3 = -4/9$

$a_4 = 1/4$

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Panel 9

http://www.mathcs.org/java/programs/FFT/index.html

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Panel 10

More abstract Setting

f a function on $[-\pi, \pi]$

$$f = \sum_{j=0}^{\infty} a_j e_j + b_j \tilde{e}_j \qquad e_j = \cos(jx), \quad \tilde{e}_j = \sin(jx)$$

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f g \, dt$$

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad x = 5e_1 + 6e_2 + 7e_3$$

e_1, e_2, e_3 $\qquad e_1 \cdot e_1 = 1 \quad e_2 \cdot e_2 = 1 \quad e_3 \cdot e_3 = 1$

$e_1 e_2 = 0$

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Panel 11

What is the space X s.t. $\exists\, e_1, e_2, \dots, e_n$ with

$$e_j \cdot e_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$x \in X$: $x = \# e_1 + \# e_2 + \dots + \# e_n = \sum_{j=1}^{n} \# e_j$

$\Rightarrow \mathbb{R}^n$, n-dim real space!!!

$\{f : f$ has a Fourier series$\}$ is like $\mathbb{R}^\infty$

and $\underbrace{\sin(nx)}_{e_n}$, $\underbrace{\cos(nx)}_{\tilde{e}_n}$ form an orthonormal basis,

i.e.

$$f = \sum^{\infty} a_j e_j + b_j \tilde{e}_j$$

∞-dim. vector space!

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Panel 12

Metric Spaces

<u>Def.</u> A metric space is a pair (X, ρ) where X is a set and ρ is a distance (or metric) function s.t.

(1) $\rho : X \times X \to \mathbb{R}^+$, i.e. $\rho(x,y)$ is non-neg., real #

(2) $\rho(x,y) = 0 \Rightarrow x = y$

(3) $\rho(x,y) = \rho(y,x)$

(4) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ (triangle inequality)

X is space, $x \in X$ are points in X.

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Panel 13

Ex: $(\mathbb{R}, \rho)$: $\rho(x,y) = |x-y|$ is $\mathbb{R}^1 = \mathbb{R}^n$

$(\mathbb{R}^n, \rho)$: $\rho(x,y) = \rho((x_1, \dots x_n), (y_1, \dots y_n)) =$ $\mathbb{R}^n_2$

$$= \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}$$

is n-dim. Euclidean space (we know and love).

Proof. (1) ✓ (2) ✓ (3) ✓ triangle?

$$\sqrt{\sum_{k=1}^{n} (x_k - z_k)^2} \leq \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2} + \sqrt{\sum_{k=1}^{n} (y_k - z_k)^2}$$

to show!

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Panel 14

Lemma: (Cauchy-Schwarz Inequality)

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right) \cdot \left(\sum_{k=1}^{n} b_k^2\right)$$

e.g. for $n=2$: $(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)$ Why?

$$\cancel{a_1^2 b_1^2} + 2 a_1 a_2 b_1 b_2 + \cancel{a_2^2 b_2^2} \leq \cancel{a_1^2 b_1^2} + a_1^2 b_2^2 + a_2^2 b_1^2 + \cancel{a_2^2 b_2^2}$$

$$0 \leq a_1^2 b_2^2 - a_1 a_2 b_1 b_2 + a_2^2 b_1^2 - a_1 a_2 b_1 b_2$$

$$a_1 b_2 (a_1 b_2 - a_2 b_1) - a_2 b_1 (a_1 b_2 - a_2 b_1) =$$

$$= (a_1 b_2 - a_2 b_1)^2 \geq 0$$

$\mathbb{R}^n$ is HW

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Panel 15

$$\sqrt{\sum_{k=1}^{n}(x_k-z_k)^2} \le \sqrt{\sum_{k=1}^{n}(x_k-y_k)^2}+\sqrt{\sum_{k=1}^{n}(y_k-z_k)^2}$$

$x_k - y_k = a_k \qquad y_k - z_k = b_k$

$$\sqrt{\sum_{k=1}^{n}(a_k+b_k)^2} \le \sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}$$

$$\sum (a_k+b_k)^2 = \sum a_k^2 + 2\sum a_k b_k + \sum b_k^2$$

$$\le \sum a_k^2 + 2\sqrt{\sum a_k}\sqrt{b_k} + \sum b_k^2$$

$$\left(\sqrt{\sum a_k^2} + \sqrt{\sum b_k^2}\right)^2 \qquad \Big/\sqrt{}$$

done

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Panel 16

HW: Prove Cauchy-Schwarz for $\mathbb{R}^3$

Ex: $(\mathbb{R}^n, \rho)$, $\rho(x,y) = \sum_{k=1}^{n} |x_k - y_k|$

$\mathbb{R}^n_1$

Ex: $(\mathbb{R}^n, \rho)$, $\rho(x,y) = \max_{1\le k\le n}(|x_k-y_k|)$

$\mathbb{R}^n_0$

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Panel 17

Ex: Draw "unit ball" in $\mathbb{R}^2_2$, $\mathbb{R}^2_1$, $\mathbb{R}^2_0$

$\rho(x,0) \le 1$

HW

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