**Example 7.4.6: Is the function *f(x) = x* Lebesgue integrable over *[0, 1]*? If so, find the integral. *A “fill in the blanks” proof:***

Before we get started, write down what it means for a function f(x) to be Lebesgue integrable:

For practice, contrast that with what it means for a function f(x) to be Riemann integrable:

Now we know what we have to proof, so let’s go. We know that *|f(x)| http://www.mathcs.org/analysis/reals/symbols/le.gif 1* over the interval *[0,1]*. The essence of this proves comes from defining the following sets:

*E1 = { x http://www.mathcs.org/analysis/reals/symbols/element.gif [0, 1]: 0 http://www.mathcs.org/analysis/reals/symbols/le.gif f(x) < 1/n }   
E2 = { x http://www.mathcs.org/analysis/reals/symbols/element.gif [0, 1]: 1/n http://www.mathcs.org/analysis/reals/symbols/le.gif f(x) < 2/n }   
E3 = { x http://www.mathcs.org/analysis/reals/symbols/element.gif [0, 1]: 2/n http://www.mathcs.org/analysis/reals/symbols/le.gif f(x) < 3/n }   
...*

And in general

*Ej = { x http://www.mathcs.org/analysis/reals/symbols/element.gif [0, 1]: \_\_\_\_\_\_\_\_\_\_\_\_\_\_ }* for *j = 1, 2, ..., n*.

The sets *Ej* are measurable and their measure is finite. Really? Why?

They are disjoint. Really? Why?

Their union (over the *j*'s) equals *[0, 1]*. Really? Actually no, the union equals \_\_\_\_\_\_\_\_ , but that does not matter. It doesn’t? Why?

Now define two functions

*Sn(x) =*

*sn(x) =*

They are simple functions. Why (make sure to lookup the definition of simple function)?

Fix an integer *n* and take a number *x* in *[0, 1)*. Then *x* must be contained in exactly one set ***E****j* because \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_. Then we have:

*sn(x) =*

*Sn(x) =*

and thus

*sn(x) http://www.mathcs.org/analysis/reals/symbols/le.gif f(x) http://www.mathcs.org/analysis/reals/symbols/le.gif  Sn(x)*

Therefore, on all of *[0, 1)*, we know that *sn(x) http://www.mathcs.org/analysis/reals/symbols/le.gif f(x) http://www.mathcs.org/analysis/reals/symbols/le.gif Sn(x)*. Why? \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_   
But then

*I\*(f)L http://www.mathcs.org/analysis/reals/symbols/le.gif or ???*

*I\*(f)L http://www.mathcs.org/analysis/reals/symbols/ge.gif  or ???*

Therefore

*I\*(f)L - I\*(f)L http://www.mathcs.org/analysis/reals/symbols/le.gif  \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_*

Since *n* was arbitrary, the upper and lower Lebesgue integrals must agree, hence the function *f* is integrable.

q.e.d.

Now **turn that example into a theorem** by checking what properties of f are the ***essential*** ones that make the proof work and stating the theorem for functions with that property or properties.