Panel 1

Last Time

Characteristic function: $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{else} \end{cases}$

Simple function: $s(x) = \sum_{j=1}^{n} c_j \chi_{A_j}(x)$, A_j are mble

Integral of simple function:

$$\int s(x)\,d\mu = \sum_{j=1}^{n} c_j \mu(A_j) \qquad \text{L.I. of simple function}$$

Integral of a bounded function:

$I^*(f) = \inf\{\int s(x)\,dx,\ s \geq f\}$

$I_*(f) = \sup\{\int s(x)\,dx,\ s \leq f\}$

If $I^*(f) = I_*(f)$ then f is Lebesgue integrable

$$\int f\,d\mu = I^*(f)$$

Panel 2

Ex: $\int_{[0,2]\cap\mathbb{Q}} x^2\,d\mu = 0$ Could not find $\int f\,dx$, i.e. Riemann integrals only over intervals

Let $s(x) = 0 \cdot \chi_{[0,2]\cap\mathbb{Q}}$
$S(x) = 4\,\chi_{[0,2]\cap\mathbb{Q}}$ } both simple, mble

$s(x) \leq f(x) \leq S(x)$

$I^*(f) = I_*(f) = 0$

$I^*(f) \leq \int_{[0,2]\cap\mathbb{Q}} S(x)\,dx = 4\mu([0,2]\cap\mathbb{Q}) = 0$

$0 \leq I_*(f) \leq I^*(f) \leq 0$

$I_*(f) \geq \int_{[0,2]\cap\mathbb{Q}} s(x)\,dx = \mu([0,2]\cap\mathbb{Q}) = 0$

Panel 3

Theorem: f bounded, E msble with $\mu(E)=0$

Then $\int_E f(x)\,d\mu = 0$

Proof f is bounded $\rightarrow$ $m \leq f(x) \leq M$

$s(x) = m\,\chi_E(x)$ $\rightarrow$ $s(x) \leq f(x) \leq S(x)$

$S(x) = M\,\chi_E(x)$

$I^*(f) \leq \int S(x)\,d\mu = m\,\mu(E) = 0$ $\Rightarrow 0 \leq I_*(f) \leq I^*(f) \leq 0$

$I_*(f) \geq \int s(x)\,d\mu = M\,\mu(E) = 0$ $\rightarrow$ $\boxed{\text{q.e.d.}}$

3

Panel 4

It is important to know which sets are measurable:

$\mathbb{Q}$, intervals, $\mu(E)=0$, open sets, closed sets, unions and intersections of msble sets (Borel sets)

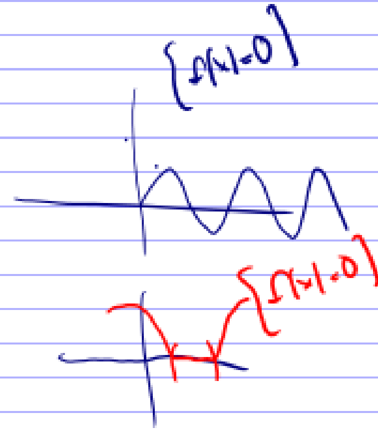
Thm: If f is some function on a measurable set then the following are equivalent.

(1) $\{x : f(x) > \alpha\}$ is msble

(2) $\{x : f(x) \geq \alpha\}$ is mble

(3) $\{x : f(x) < \alpha\}$ is mble

(4) $\{x : f(x) \leq \alpha\}$ is mble



Implies: $\{x : f(x) = \alpha\}$ is mble

4

Panel 5

Proof:

(1)⇔(4) because complements of mble sets are mble
(2)⇔(3) same

(1)⇒(2): $\{x: f(x) \geq \alpha\} = \bigcap \{x: f(x) > \alpha - \frac{1}{n}\}$

(2)⇒(1): $\{x: f(x) > \alpha\} = \bigcup \{x: f(x) \geq \alpha + \frac{1}{n}\}$

$\{x: f(x) = \alpha\} = \{x: f(x) \leq \alpha\} \cap \{x: f(x) \geq \alpha\}$

q.e.d.

5

Panel 6

Thm: If f is continuous on $\mathbb{R}$, then
$f^{-1}(-\infty, \alpha) = \{x: f(x) < \alpha\}$,
$f^{-1}((-\infty, \alpha])$, $f^{-1}(\alpha, \infty)$ $f^{-1}([\alpha, \infty))$
are mble.

Recall: f is continuous ⇔ f^{-1} of every open set is open
⇒ mble ⇒ Done

Corollary: $f^{-1}((\alpha, \beta])$, $f^{-1}((\alpha, \beta))$, $f^{-1}([\alpha, \beta))$
$f^{-1}([\alpha, \beta])$ are all mble!

#

6

Panel 7

Ex: $\int_{[0,1]} x\,dx =$

We have shown before that the function $f(x) = x^2$ is Riemann integrable. Since step functions are special cases of simple functions, we could try to replicate that proof here, suitably modified for the function $f(x) = x$. But using simple functions instead of step functions actually simplifies the proof, so here we go.

We know that $| f(x) | \leq 1$ over the interval $[0, 1]$. Define sets:

$E_1 = [x \in [0, 1]: 0 \leq f(x) < 1/n]$
$E_2 = [x \in [0, 1]: 1/n \leq f(x) < 2/n]$
$E_3 = [x \in [0, 1]: 2/n \leq f(x) < 3/n]$
...
$E_j = [x \in [0, 1]: (j-1)/n \leq f(x) < j/n]$

for $j = 1, 2, \ldots, n$. Because f is continuous, the sets E_j are measurable (really - why?), they are disjoint, and their union (over the j's) equals $[0, 1]$ (actually, the union equals $[0, 1)$, but that does not matter - why?).

Now define two simple functions

$$S_n(x) = \sum_{j=1}^{n} j/n\, X_{E_j}(x)$$
$$s_n(x) = \sum_{j=1}^{n} (j-1)/n\, X_{E_j}(x)$$

Fix an integer n and take a number x in $[0, 1)$. Then x must be contained in exactly one set E_j, and on that set we have

$$s_n(x) = (j-1)/n \leq f(x) < j/n = S_n(x)$$

Therefore, on all of $[0, 1]$, we know that

$$s_n(x) \leq f(x) \leq S_n(x)$$

7

Panel 8

Ex: $\int_{[0,1]} x\,d\mu = 1/2$

Pick some n

Proof: Note: $|f(x)| \leq 1$ on $[0,1]$. Define

$$E_1 = \{x \in [0,1]: 0 \leq f(x) < \tfrac{1}{n}\} = f^{-1}[0, \tfrac{1}{n})$$
$$E_2 = \{x \in [0,1]: \tfrac{1}{n} \leq f(x) < \tfrac{2}{n}\} = f^{-1}[\tfrac{1}{n}, \tfrac{2}{n})$$
$$\vdots$$
$$E_j = \{x \in [0,1]: \tfrac{j-1}{n} \leq f(x) < \tfrac{j}{n}\} \quad j = 1, \ldots, n$$

$\Rightarrow E_j$ are measurable because f is continuous.

$$S(x) = \sum_{j=1}^{n} j/n\, X_{E_j}(x) \text{ and } s(x) = \sum_{j=1}^{n} \tfrac{j-1}{n}\, X_{E_j}(x)$$

8

Panel 9

$E_j = \{x \in [0,1) : \frac{j-1}{n} \le f(x) < \frac{j}{n}\}$ $\quad j = 1, \ldots, n$

$\Rightarrow E_j$ are measurable because f is continuous.

$S(x) = \sum_{j=1}^{n} j/n\, \chi_{E_j}(x)$ and $s(x) = \sum_{j=1}^{n} \frac{j-1}{n} \chi_{E_j}(x)$

For a fixed n, take any $x \in [0,1)$. Then x must be in exactly one E_j

$\Rightarrow s(x) = \frac{j-1}{n} \le f(x) < \frac{j}{n} = S(x)$

$\Rightarrow s(x) \le f(x) \le S(x) \quad \forall x \in [0,1)$

$\Rightarrow I^*(f) \le \int S(x)dx = \frac{1}{n} \sum_{j=1}^{n} j\, m(E_j)$

9

Panel 10

$\Rightarrow s(x) = \frac{j-1}{n} \le f(x) < \frac{j}{n} = S(x)$

$\Rightarrow s(x) \le f(x) \le S(x) \quad \forall x \in [0,1)$

$\Rightarrow I^*(f) \le \int S(x)dx = \frac{1}{n} \sum_{j=1}^{n} j\, \mu(E_j)$

$I_*(f) \ge \int s(x)dx = \frac{1}{n} \sum_{j=1}^{n} (j-1)\, \mu(E_j)$

$0 \le I^*(f) - I_*(f) \le \frac{1}{n} \sum j\, \mu(E_j) - \frac{1}{n} \sum (j-1)\mu(E_j) =$

$= \frac{1}{n} \sum (j-(j-1))\mu(E_j) = \frac{1}{n} \sum \mu(E_j) =$

$= \frac{1}{n} \cdot 1 \quad \forall n \qquad \Rightarrow I^*(f) = I_*(f)$

because $\bigcup E_j = [0,1)$

10

Panel 11

Thus: $f(x)=x$ is Leb. intble! HW

This proof "kinda" works for any cont. function!!!

Cont: $\mu(E_j) = \mu\{x : \frac{j-1}{n} \leq f(x) \leq \frac{j}{n}\} =$

$= \mu\{x : \frac{j-1}{n} \leq x \leq \frac{j}{n}\} =$

$= \mu([\frac{j-1}{n}, \frac{j}{n})) = 1/n$ $\quad \sum (j-1)\mu(E_j)$

$\int f(x)\,d\mu = \int x\,d\mu = \lim_{n\to\infty} \frac{1}{n} \sum j\,\mu(E_j) =$

$\frac{1}{n}\sum j \frac{1}{n} = \frac{1}{n^2}\sum_{j=1}^{n} j = \frac{1}{n^2}\frac{n(n+1)}{2}$

$\to \underline{\underline{1/2}}$

11

Panel 12

Thm: If f is bounded on $[a,b]$, Riemann integrable, then f is Lebesgue intble and

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,d\mu$$

Proof: Recall: $I^*(f)_R$ was inf over all upper sums!

$I_*(f)_R$ was sup over all lower sums!

Each step function is simple, and upper sum $\geq f$

lower sum $\leq f$

$$I_*(f)_R \leq I_*(f)_L \leq I^*(f)_L \leq I^*(f)_R \Rightarrow I_*(f)_L = I^*(f)_L$$

same

12