Panel 1

Quiz #2 4 of these

① Define $\mu^*(A)$ for any $A \subset \mathbb{R}$

② Find $\mu^*(\{42\})$ (with justification)

③ Compute $\mu^*(\{1,2,3,4,5\})$

④ Prove that $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$

⑤ For any set A, define $k + A = \{k + x : x \in A\}$
Prove that $\mu^*(k+A) = \mu^*(A)$ (translation invariant)

1

Panel 2

$E \subset \mathbb{R}$ mble: $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Properties of Lebesgue Measure

(1) All intervals are mble, measure is their length

(2) Any open and any closed set is mble

(3) Union and intersections of mble sets are mble

(4) If $A = \cup A_n$ is mble $\mu(\cup A_i) \leq \sum \mu(A_i)$

(5) If $A = \cup A_n$ and A_n are disjoint and mble then

$$\mu(\cup_n A) = \sum_{n=1}^{\infty} \mu(A_i)$$

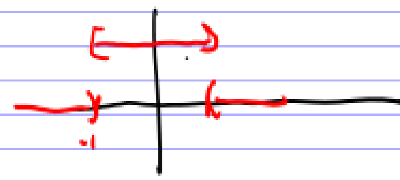
2

Panel 3

Characteristic function of a set E

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{else} \end{cases}$$

Ex: $\chi_{[-1,1]}(x)$



Dirichlet Function

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & \text{else} \end{cases} \to \chi_{\mathbb{Q}}(x)$$

3

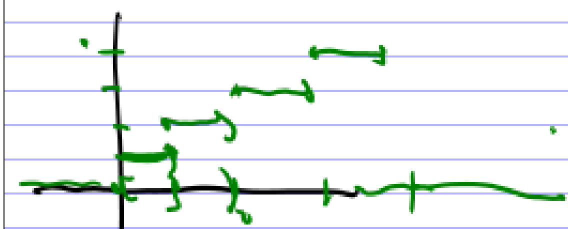
Panel 4

Simple Function:

$$s(x) = \sum_{j=1}^{n} c_j \chi_{A_j}(x) \quad , \ A_j \text{ are measurable!}$$

Ex: $s(x) = \sum_{j=1}^{4} j\, \chi_{[j-1,j)}(x)$

Ex: $s(x) = \sum_{j=1}^{4} j\, \chi_{[j-1,j)}(x) = \chi_{[0,1)}(x) + 2\chi_{[1,2)} + 3\chi_{[2,3)} + 4\chi_{[3,4)}(x).$



Also: $s(x) = \chi_{\mathbb{Q}}(x) + \chi_C(x)$

C = Cantor Set.

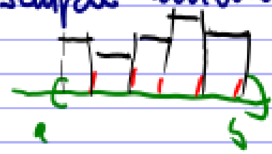
4

Panel 5

Note: (1) Is every step function simple?

(2) Is every simple function a step function?

(3) Is a simple function unique? No!

Step function (1)  Yes!

(2) $s(x) = \chi_B(x) + \chi_C(x)$ $\quad \overbrace{\chi_A(x) + \chi_B(x)}^{s_1(x)} =$

$A = [0,1), \quad B = [1,2)$

$A' = A \cap \mathbb{Q}, \quad A'' = A \cap \mathbb{I}$

$B' = B \cap \mathbb{Q}, \quad B'' = B \cap \mathbb{I}$

$$\underbrace{\chi_{A'}(x) + \chi_{A''}(x) + \chi_{B'}(x) + \chi_{B''}(x)}_{s_2(x)}$$

5

Panel 6

Canonical representation of Simple Functions

$$s(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x)$$

$\Rightarrow$ $s(x)$ has only finitely many values, call them $a_1, a_2, \dots, a_n$

$$s(x) = \sum_{j=1}^{n} a_j \chi_{\{s(x) = a_j\}}(x)$$

Thm: s any simple function. WOLOG $a_j \neq 0$, E_j are disjoint.

6

Panel 7

Find a formula for $\chi_{A\cap B}(x)$

$$\chi_{A\cap B}(x) = \chi_A(x) \cdot \chi_B(x)$$

Is $\chi_{A\cup B}(x) = \underbrace{\chi_A(x) + \chi_B(x)}_{\text{false!}} - \chi_{A\cap B}(x)$ True

7

Panel 8

Def: The Lebesgue integral of a simple function is:

$$\int s(x)\,dx_L = \sum_{j=1}^{n} c_j\, \mu(E_j) \qquad (E_j \text{ msble})$$

where $s(x) = \sum_{j=1}^{n} c_j\, \chi_{E_j}(x)$

Def: If E is msble and $s(x)$ simple, then

$$\int_E s(x)\,dx = \int \chi_E(x) \cdot s(x)\,dx$$

Lebesgue int.

8

Panel 9

Find the Lebesgue integral of the constant function $f(x) = c$ over the interval $[a, b]$.

Find the Lebesgue integral of a step function, i.e. a function s such that $s(x) = c_j$ for $x_{j-1} < x < x_j$ and the $\{ x_j \}$ form a partition of $[a, b]$.

Find the Lebesgue integral of the Dirichlet function restricted to $[0, 1]$ and of the characteristic function of the Cantor middle-third set.

Define two simple functions

$$s_1(x) = 2\,X_{[0,\,2]}(x) + 4\,X_{[1,\,3]}(x)$$

$$s_2(x) = 2\,X_{[0,\,1)}(x) + 6\,X_{[1,\,2]}(x) + 4\,X_{(2,\,3]}(x)$$

DW

Show that $s_1(x) = s_2(x)$ and $\int s_1(x)\,dx = \int s_2(x)\,dx$.

We have seen before that the representation of a simple function is not unique. Show that the Lebesgue integral of a simple function is independent of its representation.

9

Panel 10

$f(x) = c \quad \forall x \in \mathbb{R}$

$s(x) = c \cdot X_{\mathbb{R}}(x)$ is simple because $\mathbb{R}$ is measble.

$$\Rightarrow \int_{[a,b]} s(x)\,dx = \int X_{[a,b]}(x) \cdot c\,X_{\mathbb{R}}(x)\,dx =$$

$$= \int c\,X_{[a,b]\cap\mathbb{R}}(x)\,dx =$$

$$= \int c\,X_{[a,b]}(x)\,dx = c\,\mu([a,b]) =$$

$$= \underline{\underline{c(b-a)}}$$

10