Panel 1

Last Time — Leb. outer measure: $\mu^*(A) = \inf\{\sum \ell(A_i)\}$

$\mu^*(\emptyset) = 0$

$\mu^*([a,b]) = \mu^*((a,b)) = b-a$

if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$

$\mu^*(\mathbb{R}) = \infty$

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Panel 2

$A = \{x\}$

$\mu^*(\{x\}) = \inf\left(\sum \ell(A_j)\right)$

Find an open cover of $\{x\}$: $(x-\varepsilon, x+\varepsilon)$

$\mu^*(\{x\}) \leq \ell(x-\varepsilon, x+\varepsilon) = 2\varepsilon$ for any $\varepsilon > 0$.

$\Rightarrow \mu^*(\{x\}) = 0$

$A = [1,10)$ $\quad (1-\varepsilon, 10+\varepsilon)$

$\Rightarrow \mu^*(A) \leq 10 + 2\varepsilon \quad \forall \varepsilon \Rightarrow \mu^*(A) \leq 10$

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Panel 3

Subadditive: $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$

~~Additive: $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$, A, B disjoint~~

$$\mu^*(A) = 0$$

$$\mu^*(A \cup B) \leq 0 + \mu^*(B)$$

$$\mu^*(A \cup B) \leq \mu^*(B)$$

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Panel 4

$S \subseteq \mathbb{R}$, S is countable

$S = \{s_1, s_2, s_3, \ldots\}$

by countable subadditivity

by previous

$\Rightarrow \mu^*(S) = \mu^*(\cup\{s_i\}) \leq \sum \mu^*(s_i) = 0$

$\Rightarrow \mu^*(S) = 0$

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Panel 5

$kA = \{kx : x \in A\}$ $\qquad A = (0,1] \Rightarrow 3A = (0,3]$

$\mu^*(A) = 1$, $\mu^*(3A) = 3\mu^*(A)$.

$\{A_i\}$ cover A iff $\{k \cdot A_i\}$ cover kA intervals

$\mu^*(kA) = \inf\{\sum \ell(kA_i)\} = \inf(\sum k\,\ell(A_i)) =$

$= k \cdot \inf(\sum \ell(A_i))$

$= k \cdot \mu^*(A)$

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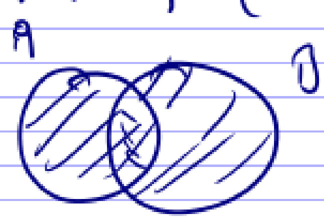
Panel 6

8) $A \Delta B = (A \cup B) \setminus (A \cap B) = (A \cap B^c) \cup (B \cap A^c)$



Claim $\mu^*(A) \le \mu^*(A \Delta B) + \mu^*(B)$

$A \subset B \cup (A \Delta B) \Rightarrow \mu^*(A) \le \mu^*(B \cup A \Delta B)$

$\le \mu^*(B) + \mu^*(A \Delta B)$ ∎

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Panel 7

Properties of Outer Measure
Lebesgue

μ also measures $\mu^*(\emptyset)=0$

(1) $\mu^*: \mathcal{P}(\mathbb{R}) \to [0,\infty)$

(2) Outer measure of any interval is its length

(3) Outer measure is subadditive, i.e.
$\mu^*(A\cup B) \le \mu^*(A)+\mu^*(B)$

(4) Outer measure is countably subadditive, i.e.
$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu^*(A_j)$.

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Panel 8

Subadditive Outer Measure.

First: $\mu^*(A\cup B) \le \mu^*(A)+\mu^*(B)$

$\mu^*(A) = \inf\{\sum \ell(A_j)\}$, A_j cover A | $\mu^*(A_n) = \inf\{\sum \ell(A_j^n)\}$
$\mu^*(B) = \inf[\sum \ell(B_j)]$ B_j cover B

$\exists$ some collection A_j, and B_j s.t.
$\sum \ell(A_j^n) \le \mu^*(A_n) + \varepsilon/2^n$
$\sum \ell(A_j) \le \mu^*(A)+\varepsilon$, $\sum \ell(B_j) \le \mu^*(B)+\varepsilon$

$\to \sum \ell(A_j) + \sum \ell(B_j) \le \mu^*(A)+\mu^*(B)+2\varepsilon \quad \forall \varepsilon$

$\sum_{n=1}^{\infty} \ell(A_j^n) \le \mu^*(A)+\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$ ε

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Panel 9

Thus: $\sum \ell(A_j) + \sum \ell(B_j) \le \mu^*(A) + \mu^*(B)$

Now: intervals $\{A_j\} \cup \{B_j\}$ cover $A \cup B$

$\mu(A \cup B) \le \sum \ell(A_j) + \sum \ell(B_j) \le \mu^*(A) + \mu^*(B)$

#

Countable subadditivity works (almost) the same!

Review the proof! In text: Prop 7.34(d)

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Panel 10

Outer measure: Any function $\mu^*: \mathcal{P}(X) \to [0, \infty)$ s.t.

(i) $\mu^*(\emptyset) = 0$

(ii) If $A \subset B$ then $\mu^*(A) \le \mu^*(B)$

(iii) μ^* is countably subadditive.

Ex: Lebesgue Outer Measure is an Outer Measure.
I.e. it is that outer measure that has added
property that $\mu^*(\text{interval}) = \text{length}$!

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Panel 11

Ex: Define $\nu^*: P(\mathbb{R}) \to [0, \infty)$ via

$$\nu^*(A) = \begin{cases} 0 & \text{if } A \text{ is finite or empty} \\ 1 & \text{if } A \text{ is infinite} \end{cases}$$

Is ν^* an outer measure?

Q. Defined for $P(\mathbb{R})$? Yes

Q. $\nu^*(\emptyset) = 0$ ✓

Q: $A \subset B$ $\begin{cases} A, B \text{ finite} \Rightarrow \nu^*(A) = 0 \le \nu^*(B) = 0 \\ A \text{ finite, } B \text{ inf} \Rightarrow \nu^*(A) = 0 \le \nu^*(B) = 1 \end{cases}$

Q: countably subadditive ✓ (try it)

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Panel 12

$$\alpha^*(A) = \begin{cases} \text{count}(A) & \text{if } A \text{ is finite} \\ \infty & \text{else} \end{cases}$$

Is this an outer Measure?

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Panel 13

(Lebesgue) Measurable Set: A set E is (Lebesgue) measurable iff for every set A one have

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \text{(by Caratheodory)}$$

$$\Rightarrow \mu(E) = \mu^*(E)$$

More obvious: 'lower measure' $\mu_e(A) = \sup(\sum \ell(A_i))$

A mble iff $\mu^*(A) = \mu_e(A)$

Note: $A \subset (A \cap E) \cup (A \cup E^c) \Rightarrow$

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cup E^c)$$

$\Rightarrow$ Mble iff $\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$

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Panel 14

Motivation: Want: A, B are disjoint:

$$\Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$$

This will be true only for mble sets!!!

Know $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$.

Want MORE: equality!

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Panel 15

Measurable Set: E is measurable if

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \quad \forall A$$

Thm: Any set with outer measure zero is msble.
and $\mu(A) = 0$

$\mu^*(E) = 0$; $A \cap E \subset E \Rightarrow \mu^*(A \cap E) \leq \mu^*(E) = 0$

$\Rightarrow \mu^*(A \cap E) = 0$

$A \cap E^c \subset A \Rightarrow \mu^*(A \cap E^c) \leq \underline{\mu^*(A)}$

$\Rightarrow \mu^*(A) \geq \mu^*(A \cap E^c) + \mu^*(A \cap E)$ $\Rightarrow E$ msble!

$+ 0$ $\Rightarrow \mu(E) = \mu^*(E) = 0$

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