Panel 1

Last Time

Open cover of a set A: collection of open sets A_j (intervals)

s.t. $\bigcup_j A_j \supset A$

Heine-Borel-Thm:

If C is compact $\subset \mathbb{R}^n$, then any open cover of C reduces to a finite subcover

Lebesgue Outer Measure: $\mu^*(A) = \inf \{ \sum \ell(A_j) : A_j$ are open (intervals) covering of $A \}$

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Panel 2

Ex: Consider $\{ (\frac{1}{n}, 1) \}$.

- is it an open cover of $(0,1)$? YES
- can you reduce it to a finite subcover?

$(\frac{1}{1000}, 1), (\frac{1}{1001}, 1) \ldots$ is a subcover, but I can't reduce it to a finite subcover

Consider $\{ (-\frac{1}{n}, 1+\frac{1}{n}) \}$

- is it an open cover of $[0,1]$? YES
- can you reduce it to a finite subcover? YES

pick $(-\frac{1}{100}, 1+\frac{1}{100})$ or any A_j. It covers $[0,1]$

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Panel 3

Ex: $\mu^*(\emptyset) = 0$

$\inf\{\sum \ell(A_j) : A_j \text{ cover } \emptyset\}$

$(0,1)$ covers $\emptyset$ $\Rightarrow \mu^*(\emptyset) \le 1$

$(0,\frac{1}{2})$ cover $\emptyset$ $\Rightarrow \mu^*(\emptyset) \le \frac{1}{2}$

$\vdots$

$(0,\frac{1}{n})$ cover $\emptyset$ $\forall n$ $\Rightarrow \mu^*(\emptyset) \le \frac{1}{n}, \forall n$

$\Rightarrow \underline{\mu^*(\emptyset) = 0}$

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Panel 4

Ex: $\mu^*(\mathbb{R}) = \infty$?

Try 1: Take any interval cover of $\mathbb{R}$, say $\{A_j\}$

$\Rightarrow$ some set must include 1 $\Rightarrow$ say A_1

some set must include 2 $\Rightarrow$ say A_2

$\vdots$

each A_j is open interval $\Rightarrow$ has finite, non-zero length

$\Rightarrow \sum \ell(A_j) = \infty$

wrong $\Rightarrow \mu^*(\mathbb{R}) = \infty$

$A_1 = (1-\frac{1}{2}, 1+\frac{1}{2})$, $A_2 = (2-\frac{1}{2^2}, 2+\frac{1}{2^2})$, ... $A_n = (n-\frac{1}{2^n}, n+\frac{1}{2^n})$

$\Rightarrow \sum \ell(A_j) = \sum_0^\infty \frac{1}{2^n} = \frac{1}{1-\frac{1}{2}} = 2 < \infty$

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Panel 5

Ex: Still working out $\mu^*(\mathbb{R})$ postpone!

Prop: Show that if $A \subset B$ then $\mu^*(A) \le \mu^*(B)$

Proof: Take any cover of ~~A~~ B $\Rightarrow$ it also covers ~~B~~ A

$\inf\{$ all covers of $A\} \le \inf\{$covers of $B\}$
$\inf$ over more stuff $\le \inf\{$ over stuff

$\mu^*(A) \le \mu^*(B)$

μ^* is monotone

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Panel 6

Thm: $\mu^*([a,b]) = b-a$, i.e. outer measure of closed interval is its length

Proof: take $(a-\varepsilon, b+\varepsilon)$ covers $[a,b]$
$\Rightarrow \mu^*([a,b]) \le \ell(a-\varepsilon, b+\varepsilon) = b-a+2\varepsilon \quad \forall \varepsilon$

$\Rightarrow \mu^*([a,b]) \le b-a$

Next: take any covering of $[a,b]$ by open intervals.
Since $[a,b]$ is compact, can extract finite subcover

$\Rightarrow \bigcup_{j=1}^{N} I_j \supset [a,b]$ Reorder I_j as follows

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Panel 7

Thus: $[a,b] \subset \bigcup_{i=1}^{N} I_n$

$I_1 = (a_1, b_1)$ is that interval containing a.

If $b_1 < b$: pick $I_2 = (a_2, b_2)$ which contains b_1

If $b_2 < b$: pick $I_3 = (a_3, b_3)$ which contains b_2

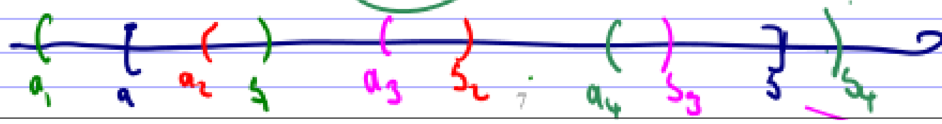
one more: if then $b_k > b \Rightarrow$ done covering $[a,b]$ NOW.

$\sum_{j=1}^{K} \ell(I_j) =$, k is last step

$= b_1 - a_1 + b_2 - a_2 + b_3 - a_3 + \dots + b_k - a_k =$

$= -a_1 + b_1 - a_2 + b_2 - a_3 + \dots + b_k$

$> -a_1 + b_k > b - a \quad \Rightarrow \mu^*([a,b]) \geq b - a$



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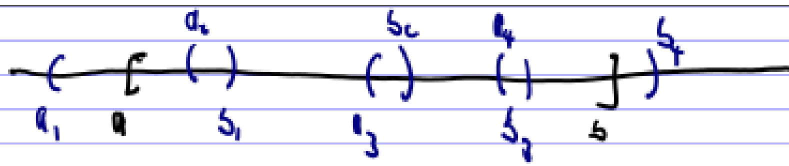
Panel 8



$\Rightarrow \mu^*([a,b]) = b - a$ #

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Panel 9

<u>Know:</u> μ^* is monotone

$$\mu^*([a,b]) = b-a$$

Prove that $\mu^*(\mathbb{R}) = \infty$

$[0,n] \subset \mathbb{R}$

$\Rightarrow \mu^*([0,n]) \leq \mu^*(\mathbb{R})$

$n \leq \mu^*(\mathbb{R}) \quad \forall n$

$\Rightarrow \mu^*(\mathbb{R}) = \infty$!

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Panel 10

<u>Prop:</u> $\mu^*((a,b)) = b-a$

$J = (a,b)$, $J_n = [a+\frac{1}{n}, b-\frac{1}{n}]$

$J_n \subset J$, $\mu^*(J_n) = b - \frac{1}{n} - (a+\frac{1}{n}) = b-a-2/n$

$J_n \subset J \subset \bar{J}$ = closure of J = $[a,b]$

$\mu^*(J_n) \leq \mu^*(J) \leq \mu^*(\bar{J})$

$b-a-\frac{2}{n} \leq \mu^*(J) \leq b-a \quad \forall n$

$\Rightarrow \mu^*(J) = b-a$

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Panel 11

Ex: $\mu^*(\mathbb{Q}\cap[0,1]) \overset{!}{=} 0$

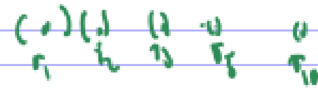
$\mathbb{Q}\cap[0,1]$ is countable $\Rightarrow$ list it in order

$\{r_1, r_2, r_3, r_4, \dots\}$

$I_n = \left(r_n - \frac{\varepsilon}{2^{n+2}}, r_n + \frac{\varepsilon}{2^{n+2}}\right)$ $\quad \mu^*(\mathbb{Q}\cap[0,1]) \le \varepsilon \ \forall \varepsilon$

$\bigcup I_n$ cover $\mathbb{Q}\cap[0,1]$

$\sum \ell(I_n) = \sum_{n=0}^{\infty} \frac{2\varepsilon}{2^{n+2}} =$



$= \varepsilon\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8} \dots\right) = \varepsilon \frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{4}+\dots\right) = \varepsilon \frac{1}{2} \frac{1}{1-\frac{1}{2}} = \varepsilon$

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Panel 12

Properties of Outer Measure

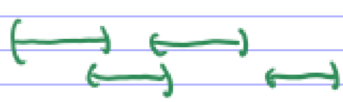
$\mathcal{P}(\mathbb{R})$ = set of all subsets

(1) $\mu^*: \mathcal{P}(\mathbb{R}) \to \mathbb{R}_0^+$

(2) Outer measure of any interval is its length

(3) Outer measure is countably subadditive, i.e.

$$\mu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \le \sum_{j=1}^{\infty} \mu^*(A_j)$$



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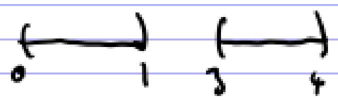
Panel 13

Subadditive. $\mu^*(A \cup B) \leq \mu^*(A) + \mu^*(B)$ ✓

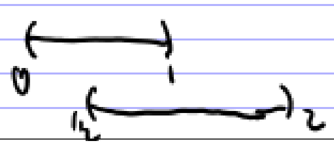
Additive: $\mu(A \cup B) = \mu(A) + \mu(B)$ (*) , $A \cap B = \emptyset$

countably subadditive: $\mu(\bigcup A_i) \leq \sum \mu(A_i)$ ✓

countably additive: $\mu(\bigcup A_i) = \sum \mu(A_i)$ (*) , A_i disjoint



has length 1+2



has length $2 \neq 25$

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Panel 14

Lebesgue Outer Measure:

$$\mu^*(A) = \inf\left(\sum \ell(A_i)\right).$$

Hausdorff Outer Measure:

$$H^d_\delta(A) = \inf\left\{\sum \ell(A_i) : A_i \text{ cover } A,\ \ell(A_i) < \delta\right\}$$

$$H^d(A) = \sup\left\{H^d_\delta(A) : \delta > 0\right\}$$

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