Panel 1

About Measure

Lebesgue Outer Measure: $\mu^*(A) = \sup\{\sum_i \ell(A_i)\}$ where

sup is over all collections of open intervals A_j s.t. $A \subset \bigcup_i A_i$ (collection A_j is an open cover of A),

$\ell(A_j)$ = length interval A_j!

Stamp! What is open cover?

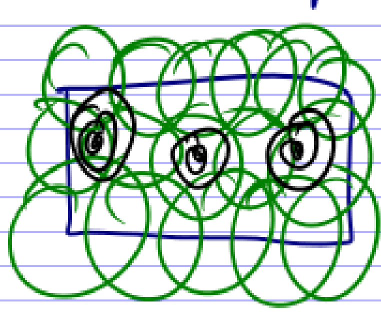
1

Panel 2

Def: For any set A we say that a collection of open sets $\{A_\alpha\}$ is an open cover of A. If

$$A \subset \bigcup_\alpha A_\alpha$$

Thm: If $C \subset \mathbb{R}$ compact then every open cover of C has finite subcover of C.

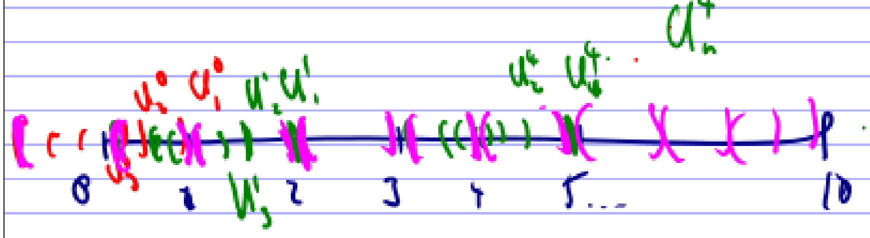


throw away black circles
$\Rightarrow$ still have cover C.

2

Panel 3

Ex: Consider $[0,10]$ and $U_n^j = \{|x-j|<\frac{1}{n}\}$. Then all U_n^j, $j=0,1,\dots,10$, and $n=0,1,\dots$, cover $[0,10]$.



$U_n^0 = \{|x-0|<\frac{1}{n}\} = \{|x|<\frac{1}{n}\}$ $\quad [0,10] \subset \bigcup_{j,k} U_k^j$

$U_n^1 = \{|x-1|<\frac{1}{n}\}$ $\quad$ Enough to take

$U_n^4 = \{|x-4|<\frac{1}{n}\}$

U_1^j, $j=0,1,\dots,10$ is finite subcover!

3

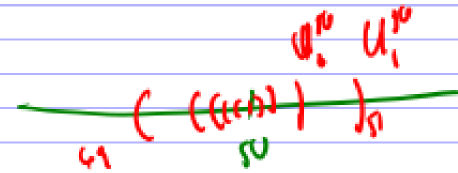
Panel 4

Ex: Consider $[0,\infty)$ and $U_n^j = \{|x-j|<\frac{1}{n}\}$ $\forall j, \forall n$. All U_n^j cover $[0,\infty)$

Can not extract finite subcover. Say there is a finite subcover. Pick largest j.

$x = j+4 \notin \bigcup_{\text{finite}} U_n^j$

Say largest $j=50$: U_n^{50} =



4

Panel 5

Ex: Consider $(0,1)$. Show that not every open cover has a finite subcover.

There is one cover for which I can't find finite subcover

1. $U_n = (-n, n)$? Is cover, can easily extract finite subcover
2. $U_n = (\frac{1}{n}, 1)$? $\bigcup_{n=2}^{\infty} (\frac{1}{n}, 1) = (0,1)$

$\cancel{(\frac{1}{2},1)} \cup \cancel{(\frac{1}{3},1)} \cup \cancel{(\frac{1}{4},1)} \cup \ldots$

Suppose $\bigcup_{n=2}^{N} (\frac{1}{n}, 1)$. This does not include $\frac{1}{N+1} \in (0,1)$

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Panel 6

Thus: $\mu^*(A) = \inf \{ \sum \ell(A_i),$ all open covers of $A \}$

Every open set in $\mathbb{R}$ is countable union of disjoint open intervals

$\Rightarrow \mu^*(A) = \inf \{ \sum \ell(A_i),$ all coverings of open intervals of $A \}$

Ex: Find $\mu^*(\emptyset)$ for empty set $\emptyset$

$\mu^*(\emptyset) = 0$

6

Panel 7

Ex: $\mu^*(\mathbb{R}) = \infty$. Why?

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