Panel 1

Review of Taylor Series: $f \in C^{n+1}([a,b])$. Then

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_{n+1}(x)$$

$$R_{n+1}(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n)}(t)\,dt =$$

$$= \frac{f^{(n+1)}(d)}{(n+1)!}(x-c)^{n+1} \quad , d \in [c,x]$$

Not every C^∞-function equals its Taylor series.
Those who do are real analytic.

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Panel 2

Techniques for finding Taylor Series:

1. Taylor's formula $a_n = \frac{f^{(n)}(c)}{n!}$
2. Subst. into known formulas
3. Differentiation
4. Integration
5. Mult.
6. Division

Need: e^x, $\sin(x)$, $\cos(x)$, $\frac{1}{1-x}$

$\sin(2x)e^{-x^2}$

$1/(1-x^2)$ (2)

(3) $\frac{1}{(1-x)^2} = \frac{d}{dx}(1-x)^{-1} =$

(2) e^{5x^2}

(2),(5) $x^2 e^{2x}$

(6) $\tan(x)$

(2)(3) $\frac{2x}{(1-x^2)^2} = \frac{d}{dx}\frac{1}{1-x^2}$

$\lim_{x \to 0} \frac{\sin(x)}{x}$

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Panel 3

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) =$$

$$= \lim_{x\to 0} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots = 1$$

$\tan(x) = (\sin(x))/(\cos(x)) =$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) \div \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) = x + x^3\left(\frac{1}{2!} - \frac{1}{3!}\right)$$

$$- x + \frac{x^3}{2!} - x^5/4! + \dots$$

$$x^3\left(\frac{1}{2!} - \frac{1}{3!}\right) - x^5\left(\frac{1}{4!} - \frac{1}{5!}\right) + \dots$$

$\tan(0) = 0$

$\tan'(x)\big|_0 =$

$$\frac{f'''(0)}{3!} = \frac{1}{2!} - \frac{1}{3!}$$

$$f'''(0) = \left(\frac{1}{2} - \frac{1}{6}\right) \cdot 6 = 2$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$\frac{f'(0)}{1!} \quad \frac{f''(0)}{2!} \quad \frac{f'''(0)}{3!}$$

Panel 4

let $f(x) = \sin(2x)\, e^{-x^2}$. Find $f^{(4)}(0) = 0$

$$f^{(5)}(0) = \left(1 + \frac{8}{6} + \frac{32}{120}\right) \cdot 120$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\sin(2x) = \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots\right) \cdot \left(1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots\right)$$

$$= \text{junk} + x^4(0) \Rightarrow$$

$$+ x^5 \underbrace{\left(2 \cdot \frac{1}{2!} + \frac{8}{3!} + \frac{32}{5!}\right)}$$

good question

Panel 5

$\frac{x}{x+1}$ $\quad x+1\overline{)\,x}$ $\quad \frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{2!}-\frac{1}{3!}\right)$

$\frac{\sin}{\cos}$ $\quad \cos\overline{)\,\sin}$ $\quad x+\left(\frac{1}{2!}-\frac{1}{3!}\right)x^3+\frac{1}{2!}\left(\frac{1}{2!}-\frac{1}{3!}\right)-\left(\frac{1}{4!}-\frac{1}{5!}\right)$

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots \overline{)\,x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots}$$

$$-x+\frac{x^3}{2!}-\frac{x^5}{4!}\dots$$

$$x^3\left(\frac{1}{2!}-\frac{1}{3!}\right)-x^5\left(\frac{1}{4!}-\frac{1}{5!}\right)+x^7\left(\frac{1}{6!}-\frac{1}{7!}\right)$$

$$-x^3\left(\quad\right)+x^5\left(\frac{1}{2!}-\frac{1}{3!}\right)\frac{1}{2!}-\dots$$

$$x^5\left(\left(\frac{1}{2!}-\frac{1}{3!}\right)\frac{1}{2!}-\left(\frac{1}{4!}-\frac{1}{5!}\right)\right)$$

?

Messy

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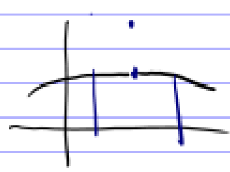
Panel 6

A new Type of Integral

Quirks of RI (Riemann Integral)

- change $f(x)$ at one point → RI does not change
  at one point to be ∞ → RI d.n.e



- there are functions that are essentially const. yet RI does not exist (Dirichlet function)
- can not $\int_C f(x)\,dx$, C - Cantor set. RI only works for intervals
- only works $f:\mathbb{R}^n\to\mathbb{R}$

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Panel 7

RI is based on concept of length of intervals.
$\Rightarrow$ to generalise integration we need to expand concept of "length"

Measure

- applies to more sets, ideally to all subsets of $\mathbb{R}$ $\quad m(A) \geq 0 \ \forall A \subset \mathbb{R}$
- similar to length for simple sets like (a,b) $\quad m((1,4)) = 3$
- length of countable many disjoint sets is sum of their lengths

$m((1,2) \cup (3,4)) = 2$, even for countable unions

impossible

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Panel 8

Two stages: (i) define a function that applies to all sets
(ii) Restrict that function to get properties of length

Def: Lebesgue Outer Measure of a subset $A \subset \mathbb{R}$ is

$$\mu^*(A) = \inf \left\{ \sum \ell(A_i) \right\}$$

where the inf is over all collections of open intervals that cover A, i.e. $A \subset \bigcup_j A_j$, $\ell(A) = $ length

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Panel 9

Ex: $\mu^*(\emptyset)$ $\quad\quad \emptyset \subset (-\frac{1}{n}, \frac{1}{n})$

$\Rightarrow (-\frac{1}{n}, \frac{1}{n})$ covers $\emptyset$

$0 \le \mu^*(\emptyset) \le \ell(-\frac{1}{n}, \frac{1}{n}) = \frac{2}{n} \quad \forall n$

$\Rightarrow \underline{\underline{\mu^*(\emptyset) = 0}} \quad\quad (\mu^*(A) \ge 0)$

Also: $\mu^*(\emptyset) \le \ell((1,6)) = 5$

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Panel 10

Prop: If $A \subset B$ then $\mu^*(A) \le \mu^*(B)$, i.e. μ^* is monotone

[Proof] Take any cover of $B \Rightarrow$

$B \subset \bigcup I_n$ $\quad\quad$ Since $A \subset B$

I_n also cover A.

Every cover of B also covers A.

$\mu^*(A) \le \mu^*(B)$, because inf. over more stuff $\le$ inf. less stuff!

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