Panel 1

Theorem: If $f \in C^{n+1}([a,b])$ then

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \dots + \frac{f^{(n)}(c)}{n!} + R_{n+1}(x)$$

where $$R_{n+1}(x) = \frac{f^{(n+1)}(d)}{(n+1)!}(x-c)^{n+1}$$

Lagrange Form of Remainder.

1

Panel 2

Remainder: $R_{n+1}(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt$

Recall: MVT for integrals

$\circledast$ $$\int_a^b f(x)g(x)\,dx = f(d)\int_a^b g(x)\,dx \quad , d \in [a,b]$$

From calc: $\int_a^b f(x)\,dx = f(d)(b-a)$ (set $g(x)=1$)

Try $\circledast$ with $g(x) = (x-t)^n$, $f(x) = f^{(n+1)}(x)$

$$\Rightarrow \int g \cdot f = \int (x-t)^n f^{(n+1)}(x)\,dx = f^{(n+1)}(d) \int_c^x (x-t)^n\,dt$$

2

Panel 3

$$\int g \cdot f = \int (x-t)^n f^{(n+1)}(x)\,dx = f^{(n+1)}(d) \int_c^x (x-t)^n\,dt$$

$$f^{(n+1)}(d)\left[\frac{-1}{n+1}(x-t)^{n+1}\Big|_c^x\right] =$$

$$\frac{f^{(n+1)}(d)}{(n+1)}(x-c)^{n+1}$$

$$\Rightarrow \frac{1}{n!}\int_0^x (x-t)^n f^{(n+1)}(t)\,dt = \frac{1}{n!}\,\frac{f^{(n+1)}(d)}{(n+1)}(x-c)^{n+1}$$

#

3

Panel 4

Application 1: If $f \in C^{n+1}([a,b])$ and $c \in [a,b]$ then

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + r(x)(x-c)^n$$

and $\lim_{x \to c} r(x) = 0$

Proof $$R_{n+1}(x) = \underbrace{\frac{f^{(n+1)}(d)}{(n+1)!}(x-c)}_{r(x)} \cdot (x-c)^n$$

4

Panel 5

Application 2: $\lim_{n\to\infty} \sqrt{n+\sqrt{n}} - \sqrt{n} = \frac{1}{2}$

Consider $f(x) = \sqrt{1+x} \in C^2([-\frac{1}{2}, \frac{1}{2}])$

$$f(x) = f(0) + \frac{f'(0)}{1!}x + r(x)\cdot x, \quad \lim_{x\to 0} r(x) = 0$$
$$= 1 + \frac{1}{2}x + r(x)x$$

$$\sqrt{n+\sqrt{n}} = \sqrt{n\left(1+\frac{1}{\sqrt{n}}\right)} = \sqrt{n}\,\underbrace{\sqrt{1+\frac{1}{\sqrt{n}}}}\,. \quad \text{Take } x = \frac{1}{\sqrt{n}}$$

$$= \sqrt{n}\left(1 + \frac{1}{2}\frac{1}{\sqrt{n}} + r\left(\frac{1}{\sqrt{n}}\right)\frac{1}{\sqrt{n}}\right) = \sqrt{n} + \frac{1}{2} + r\left(\frac{1}{\sqrt{n}}\right)$$

$$\sqrt{n+\sqrt{n}} - \sqrt{n} = \frac{1}{2} + \cancel{r\left(\frac{1}{\sqrt{n}}\right)} \to 0 \text{ as } n\to\infty$$

Panel 6

Relativity Theory 101

Know: $p(t) = m\cdot v(t)$ momentum, m = mass const, v = velocity

Newton: $F(t) = \frac{d}{dt}p(t) = m\,v'(t) = m\,a(t)$

If $F(t) = f$ is constant:

$$f = m v'(t) \Rightarrow v'(t) = f/m \Rightarrow v(t) = f/m \cdot t + c$$

I.e. if a particle is subjected to constant force, then its velocity goes to ∞ at $t\to\infty$!

Panel 7

What if mass is not constant?

$$m = m(t) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

m_0 is mass at rest,
$c = 3 \cdot 10^8 \text{ m/sec}$ speed of light.

mass becomes ∞ if $v \to c$, if $m_0 \neq 0$
But photones have $m_0 = 0$, so they can go at $v = c$!

$$p(t) = m(t)\, v(t) = \frac{v(t)\, m_0}{\sqrt{1 - \frac{v(t)^2}{c^2}}}$$

$$F(t) = \frac{d}{dt}\left(p(t)\right) = \frac{d}{dt}\left(\frac{m_0 v(t)}{\sqrt{1 - \frac{v(t)^2}{c^2}}}\right) = -\frac{m0\left(\frac{d}{dt} v(t)\right) c^2}{\left(-c^2 + v(t)^2\right)\sqrt{-\frac{-c^2 + v(t)^2}{c^2}}}$$

7

Panel 8

$$F(t) = \frac{m_0\, v'(t)\, c^2}{\left(v(t)^2 - c^2\right)\sqrt{\frac{c^2 - v(t)^2}{c^2}}}$$

Solve for v!

This is a DE

$$v(t) = \frac{c\left(\int F(t)\,dt + D\right)}{\sqrt{\left(\int F\,dt\right)^2 + 2\left(\int F\,dt\right)D + D^2 + c^2 m_0^2}}$$

F is constant, i.e. $F = f$

$$v(t) = \frac{c\left(f \cdot t + \cancel{D}\right)}{\sqrt{(ft)^2 + \cancel{2ft D} + \cancel{D}^2 + c^2 m_0^2}}$$

D = constant
$= 0$

8

Panel 9

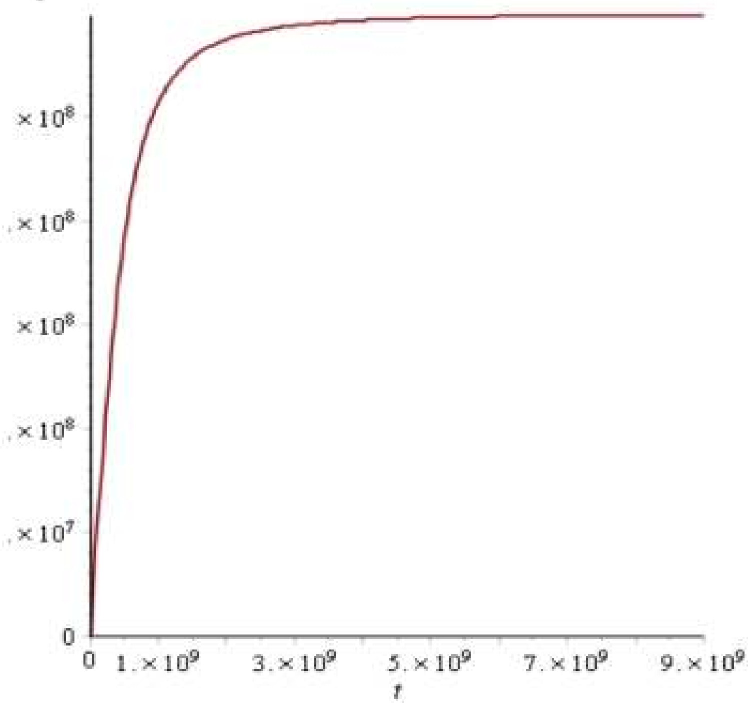
$$v(t) = \frac{c f t}{\sqrt{(ft)^2 + c^2 m_0^2}}$$ is velocity of a relativistic object, const force.

$$\lim_{t\to\infty} \frac{c f t}{\sqrt{(ft)^2 + c^2 m_0^2}} = \lim_{t\to\infty} \frac{(ft)\, c}{(ft)\sqrt{1 + \frac{c^2 m_0^2}{(ft)^2}}} = C$$

let $f = 1$, $m_0 = 1$, $C = 3 \cdot 10^8$ m/sec.

9

Panel 10



10

Panel 11

Recall $m(t) = \frac{m_0}{\sqrt{1-v^2/c^2}}$ ← Lorentz transformation

$\Rightarrow c^2 m(t) = \frac{c^2 m_0}{\sqrt{1-v^2/c^2}}$

$f(v) = \frac{c^2 m_0}{\sqrt{1-v^2/c^2}} = \underline{f(0)} + \frac{f'(0)}{1!}v + \frac{f''(0)}{2!}v^2 + \ldots$

$f(0) = \underline{c^2 m_0}$ $\qquad f'(x) = c^2 m_0 \frac{-1}{2}\left(1-v^2/c^2\right)^{-3/2}\left(\cancel{-}2v/c^2\right)$

$f'(0) = 0$ $\qquad = \frac{\cancel{c^2} m_0 v}{\left(1-v^2/c^2\right)^{3/2}}$

11

Panel 12

$f''(0) = m_0$

<u>Thus:</u> $mc^2 = m_0 c^2 + \frac{1}{2} m_0 v^2 + R(v) = E$

↑ Energy

$\Rightarrow E = m_0 c^2$ for $v = 0$

12

Panel 13

Special Relativity ✓

13

Panel 14

A new Type of Integral

Quirks of R.I.:

14