Panel 1

$f(x) = e^x \overset{?}{=} 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$a_n = \frac{f^{(n)}(0)}{n!}$ in Taylor series if f had one

Stupid argument, because

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ conv. $\forall x$ by ratio test $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{x^{n+1} n!}{(n+1)!\, x^n} = \frac{1}{n+1}|x| \to 0$

Also $\frac{f^{(n)}(0)}{n!} = a_n$ match for $f(x) = e^x$

$\Rightarrow$ ~~$e^x = \sum \frac{x^n}{n!}$~~

1

Panel 2

Find the Taylor series centered at zero for:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{e^{-1/h^2}}{h} = \lim_{h\to 0} \frac{1/h}{e^{1/h^2}}$

$= \lim_{u\to\pm\infty} \frac{u}{e^{u^2}} = 0$ by l'Hospital

$f''(0) = 0$

$\boxed{f^{(n)}(0) = 0 \ \forall n}$

① $\sum \frac{f^{(n)}(0)}{n!} x^n$ conv. $\forall x$
$\equiv 0$

② $\sum \frac{f^{(n)}(0)}{n!} x^n \neq f(x)$

2

Panel 3

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$ continued ...

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f(0)}{h} = \lim_{h \to 0} \frac{f'(h)}{h} = \lim \frac{\frac{3}{h^3} e^{-1/h^2}}{h} =$$

$$= \lim \frac{3}{h^4} e^{-1/h^2} = \lim_{h \to 0} \frac{2/h^4}{e^{1/h^2}} =$$

$$= \lim_{u \to \infty} \frac{u^4}{e^{u^2}} = 0 \text{ by l'Hospital}$$

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Panel 4

Can not argue: $f(x) = e^x$, $\frac{f^{(n)}(0)}{n!} = 1/n!$, $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ conv $\forall x$

$\cancel{\Rightarrow}$ $e^x = \sum \frac{x^n}{n!}$

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Panel 5

Taylor's Theorem: Suppose $f \in C^{n+1}([a,b])$. Then

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \dots \frac{f^{(n)}(c)}{n!}(x-c)^n + R_{n+1}(x)$$

where $$R_{n+1}(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt \quad \forall n$$

In particular if $f \in C^\infty([a,b])$ then the Taylor series of f conv. to f iff

$$\lim_{n\to\infty} R_{n+1}(x) = 0 \quad \forall x \in [a,b]$$

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Panel 6

Proof (by induction) $R_{n+1}(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt$

<u>n=0:</u> $f(x) = f(c) + \int_c^x f'(t)\,dt = f(c) + f(t)\Big|_c^x = f(x)$

true for n: $f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_{n+1}(x)$

where $R_{n+1}(x) = \frac{1}{n!}\int_c^x (x-t)^n \cdot f^{(n+1)}(t)\,dt$

By parts: $u' = (x-t)^n \qquad u = \frac{-1}{n+1}(x-t)^{n+1}$

$v = f^{(n+1)}(t) \qquad v' = f^{(n+2)}(t)$

$$\Rightarrow R_{n+1}(x) = \frac{1}{n!}\left[\frac{-1}{n+1}(x-t)^{n+1} f^{(n+1)}(t)\Big|_c^x + \frac{1}{n+1}\int (x-t)^{n+1} f^{(n+2)}(t)\,dt\right]$$

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Panel 7

$$\Rightarrow R_{n+1}(x) = \frac{1}{n!}\left[-\frac{1}{n+1}(x-t)^{n+1} f^{(n+1)}(t)\Big|_c^x + \frac{1}{n+1}\int (x-t)^{n+1} f^{(n+2)}(t)\,dt\right]$$

$$= \frac{1}{n!}\left(0 + \frac{1}{n+1}(x-c)^{n+1} f^{(n+1)}(c)\right) + \frac{1}{n+1}\int_c^x (x-t)^{n+1} f^{(n+2)}(t)\,dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + \frac{1}{(n+1)!}\int_c^x (x-t)^{n+1} f^{(n+2)}(t)\,dt$$

$$\Rightarrow f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \dots \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + R_{n+2}(x)$$

\#

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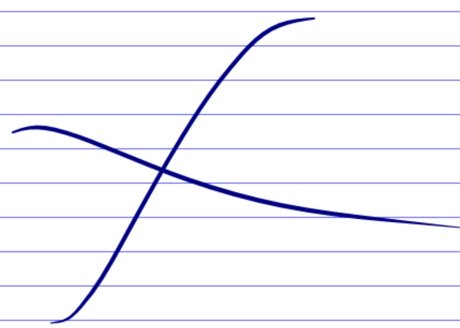
Panel 8



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Panel 9

Left to show:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \iff \lim_{n\to\infty} R_{n+1}(x) = 0$$

obvious ?

H/W

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Panel 10

Corollary: If f is C^{n+1} then f can be approximated by a polynomial of degree n.

Q: Approx $\tan(x^2+1)$ by a second-degree polyn. near the origin.

$\tan(x^2+1) \approx a_0 + a_1 x + a_2 x^2$ for $|x|$ small

$a_0 = f(0) = \tan(1) = 1.557$ $f''(0) \approx 6.851$

$a_1 = \frac{f'(0)}{1!}$: $f'(x) = \sec^2(x^2+1)\cdot 2x \Rightarrow f'(0) = 0$ /

$a_2 = \frac{f''(0)}{2!}$: $f''(x) = 2\sec(x^2+1)\cdot\sec(x^2+1)\tan(x^2+1)\cdot 2x\cdot 2x + \sec^2(x^2+1)\cdot 2$

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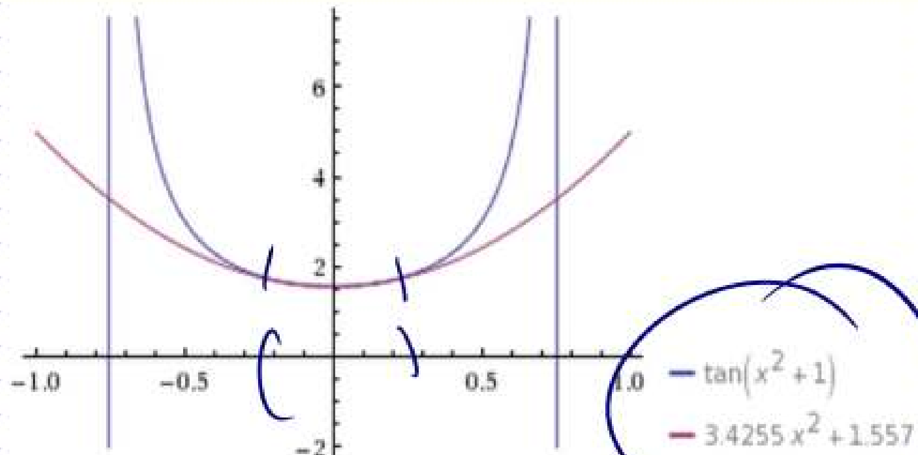
Panel 11

$\tan(x^2+1) \approx 1.557 + 0 \cdot x + \frac{6.851}{2!} x^2$



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Panel 12

$E = mc^2$ derives from such an approx!

next time

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Panel 13

Prove that:

(1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

(2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

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Panel 14

(1) $\frac{1}{1-x} = \sum x^n \quad , |x| < 1$

$1 + \cancel{x} + \cancel{x^2} + \dots + \cancel{x^n} = S_n(x)$

$\cancel{x} + \cancel{x^2} + \dots / + x^{n+1} = x S_n(x)$

$S_n(x) - x S_n(x) = 1 - x^{n+1}$

$S_n(x) = \frac{1-x^{n+1}}{1-x}$

$\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n$

$\Rightarrow \frac{1}{1-x} = \underbrace{1 + x + x^2 + \dots + x^n}_{\text{Taylor polyn.}} + \underbrace{\frac{x^{n+1}}{1-x}}$

$R_{n+1}(x) = \frac{x^{n+1}}{1-x} \to 0$ as $n \to \infty$ $\forall |x| < 1$

Thus: series $\underline{\text{is}}$ Taylor series, is equal to $\frac{1}{1-x}$.

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Panel 15

② $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $f(x) = e^x$

Proof

$$R_{n+1}(x) = \frac{1}{n!} \int_c^x (x-t)^n f^{(n+1)}(t)\,dt \quad , c=0$$

$$= \frac{1}{n!} \int_0^x (x-t)^n e^t\,dt$$

$$|R_{n+1}(x)| \leq \frac{1}{n!} e^x \left| \int_0^x (x-t)^n\,dt \right| = e^x \frac{1}{n!} \left(-\frac{1}{n+1} (x-t)^{n+1} \Big|_0^x \right) =$$

$$= e^x \frac{1}{n!} \frac{1}{(n+1)} |x^{n+1}| = e^x \frac{|x|^{n+1}}{(n+1)!} \to 0 \;\; \forall x$$

as $n \to \infty$

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Panel 16

If $f \in C^\infty([a,b])$ and f is equal to its Taylor series, then f is called real analytic

Recall from complex:

f is $\mathbb{C}$-diffble once in an open set

$\Rightarrow$ f is C^∞ and

f = Taylor series (analytic)

Real Analytic functions are the really good guys!

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Panel 17

$\ln(1-x)$ is real analytic near $x=0$.

$f(x) = \ln(1-x)$

$f'(x) = -\frac{1}{1-x} = -\sum x^n$

$f''(x) = -\frac{1}{(1-x)^2}$

$f'''(x) = -2/(1-x)^3$

$f^{(4)}(x) = -3!/(1-x)^4$

$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n}$

$\Rightarrow f^{(n)}(0) = -(n-1)!$

$\frac{f^{(n)}(0)}{n!} = -\frac{(n-1)!}{n!} = -\frac{1}{n}$

$\sum_{n=1}^{\infty} -\frac{x^n}{n} \overset{?}{=} \ln(1-x)$

$R_{n+1}(x) \to 0$

$\ln(1-x) = \int f'(x) = -\sum \int x^n dx = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \sum_{n=1}^{\infty} -\frac{x^n}{n}$

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