Panel 1

Finding Taylor Series

$$a_n = \frac{f^{(n)}(c)}{n!}$$

$$\sin(2x) = 0 + \overset{k=0}{\frac{2}{1!}}x + 0 - \overset{k=1}{\frac{8}{3!}}x^3 - 0 + \overset{k=2}{\frac{32}{5!}}x^5$$

$a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k+1}}{(2k+1)!} x^{2k+1}$$

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Panel 2

$\cos(x)$ $\qquad \cos(\pi/2) = 0$

$f'(x) = -\sin(x)$ $\qquad -\sin(\pi/2) = -1$

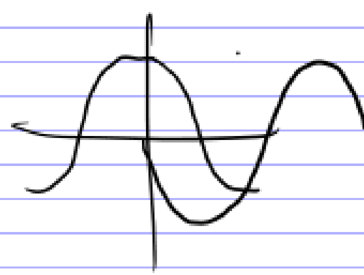
$f''(x) = -\cos(x)$ $\qquad 0$

$f'''(x) = \sin(x)$ $\qquad 1$

$f^{(4)}(x) = \cos(x)$ $\qquad 0$

$\vdots$ $\qquad -1$

$\vdots$



$$\cos(x) = 0 - \frac{1}{1!}\left(x-\frac{\pi}{2}\right) - 0 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 + 0 - \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \dots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!}\left(x-\frac{\pi}{2}\right)^{2k+1} = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(x-\frac{\pi}{2}\right)^{k}$$

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Panel 3

$f(x) = \ln(x)$ $\quad \ln(2)$ $\quad a_k = \frac{f^{(k)}(c)}{k!}$

$f'(x) = \frac{1}{x} = x^{-1}$ $\quad 1/2$ $\quad \ln(x) = \ln(2) +$

$f''(x) = -x^{-2}$ $\quad -\frac{1}{2^2}$ $\quad \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 2^k}(x-2)^k$

$f'''(x) = +2x^{-3}$ $\quad 2!/2^3$

$f^{(4)}(x) = -3!\,x^{-4}$ $\quad -3!/2^4$ $\quad \frac{(k-1)!}{k!} = \frac{1}{k}$

$f^{(5)}(x) = 4!\,x^{-5}$ $\quad 4!/2^5$

$f^{(6)}(x) = -5!\,x^{-6}$ $\quad -5!/2^6$

$$\ln(x) = \ln(2) + \frac{1}{2}(x-2) - \frac{1}{2^2 2!}(x-2)^2 + \frac{2}{3!\,2^3}(x-2)^3 - \frac{3!}{4!\,2^4}(x-2)^4$$

$$= \ln(2) + \frac{1}{2}(x-2) - \frac{1}{2 \cdot 2^2}(x-2)^2 + \frac{1}{3 \cdot 2^3}(x-2)^3 \ldots$$

Panel 4

$\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ $\quad 1$

$[\cosh(x)]' = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$ $\quad 0$

$\frac{d^2}{dx^2} \quad = \frac{1}{2}(e^x + e^{-x}) = \cosh(x)$ $\quad 1$

$\frac{d^3}{dx^3} \quad = \frac{1}{2}(e^x - e^{-x}) = \sinh(x)$ $\quad 0$

$$\cosh(x) = 1 + 0 + \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$$

Why is $\frac{1}{2}(e^x + e^{-x})$ called cosh ??

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Panel 5

$\sqrt{1+x}$, $c=0$

$\cancelto{1}{(1+x)}^{1/2}$

$f' = \frac{1}{2}\cancelto{1}{(1+x)}^{-1/2}$

$f'' = -\left(\frac{1}{2}\right)^2 \cancelto{1}{(1+x)}^{-3/2}$

$f''' = \left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)\cancelto{1}{(1+x)}^{-5/2}$

$f^{(4)} = -\left(\frac{1}{2}\right)^2\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cancelto{1}{(1+x)}^{-7/2}$

$\vdots$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2^2}x^2 + \frac{1\cdot 3}{2^3}x^3$$

$$\underbrace{1 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1\cdot 3 \cdots}{2^k} x^k}_{\text{tricky}}$$

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Panel 6

Find the Taylor series centered at zero for:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

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Panel 7

arctan(x)

$$\frac{1}{1+x^2} = (1+x^2)^{-1} = \frac{1}{1-(-x^2)} = \sum (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$-\frac{2x}{(1+x^2)^2}$$

$\vdots$ to difficult!

$$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} = \frac{d}{dx} \arctan(x) \quad \Big| \int$$

$$\int \sum_{n=0}^{\infty} (-1)^n x^{2n} \, dx = \arctan(x)$$

$$\sum_{n=0}^{\infty} (-1)^n \int x^{2n} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$$

set $x=0$ to find $C=0$

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Panel 8

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\arctan(1) = \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\pi = 4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots\right)$$

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Panel 9

$$\frac{x}{(1-x)^2} = x\frac{1}{(1-x)^2} = x\left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right) =$$

$$= x\left(\sum x^n\right)\left(\sum x^n\right) =$$

$$= x\left(1+x+x^2+\dots\right)\left(1+x+x^2+\dots\right) =$$

$$= x + 2x^2 + 3x^3 + 4x^4 \dots$$

$$= x\frac{1}{(1-x)^2} = x\cdot\left(\frac{1}{1-x}\right)' = x\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) =$$

$$= x\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=1}^{\infty} n x^n$$

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