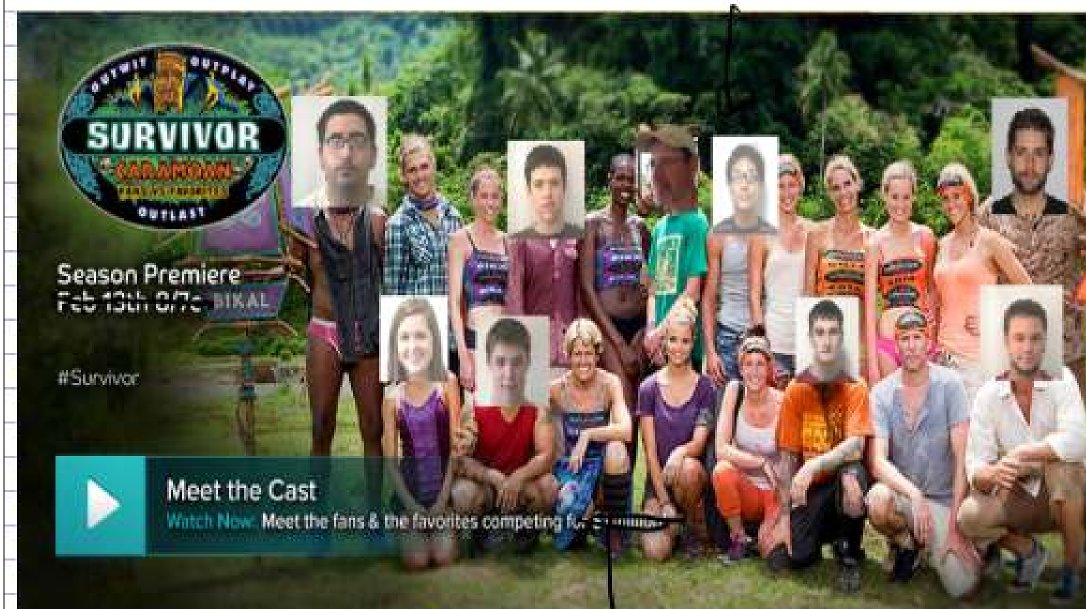
Panel 1

Welcome to ...



Mon, Jan 14 Grades!

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Panel 2

Same class as last year -
different material + better handwriting

- Taylor series	Midterm	100
- Measure Theory	Final	100
- Lebesgue integration	Quizzes	100
- Metric space	HW	100
- TBD		

Web site + ITR

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Panel 3

Power Series: $f(x) := \sum_{n=0}^{\infty} a_n (x-c)^n$

center of conv.: $x = c$

radius of conv.: $\lim \frac{a_n}{a_{n+1}} = r$ $\quad$ $\lim \frac{a_{n+1}}{a_n} < 1$

converges uniformly + absolutely $|x-c| \le \rho < r$

check endpoints

limit function is cont.

diffble and $\frac{d}{dx} \sum_0^{\infty} a_n (x-c)^n = \sum a_n n (x-c)^{n-1}$

intble and $\int \sum a_n (x-c)^n dx = \sum a_n \frac{1}{n+1} (x-c)^{n+1}$

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Panel 4

Ex: $\sum_{n=1}^{\infty} \frac{1}{n} (x-2)^n$ $\quad$ $|x-2| < 1$ ← radius (center: 2)

① $\lim \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = 1$

② $\lim \left| \frac{(x-2)^{n+1} \, 1/(n+1)}{(x-2)^n \, 1/n} \right| = |x-2| \lim \frac{1/(n+1)}{1/n} = |x-2| \cdot 1 < 1$

$\Rightarrow |x-2| < \frac{1}{1} = 1$

$x=1$: $\sum \frac{1}{n} (-1)^n$ $\quad$ alt. harm. series (conv.)

$x=3$: $\sum \frac{1}{n} 1^n = \sum \frac{1}{n}$ $\quad$ harmonic series (div)

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Panel 5

Every power series represents a C^∞ - function, i.e.
Σ inf. often diff'ble

Converse: Is every C^∞ function a power series???

Take $f(x) := \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \dots$

$f'(x) = \cancel{1} a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \dots$

$f(c) = a_0$

$f'(c) = a_1$

$f''(x) = 2a_2 + 3 \cdot 2 \cdot (x-c) + \dots$

$f''(c) = 2a_2$

$\Rightarrow f^{(n)}(c) = n! \cdot a_n$

$f'''(c) = 3! a_3$

$\Rightarrow a_n = \dfrac{f^{(n)}(c)}{n!}$

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Panel 6

If $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$, then $f^{(n)}(x_0) = n! \, a_n$

or $a_n = \dfrac{f^{(n)}(x_0)}{n!}$

Ex: Suppose $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} a_n x^n$. Find a_n

n		
0	$f(0)$	1
1	$f'(0)$	1
2	$f''(0)$	1
3	$f^{(3)}(0)$	1

$f'(x) = +1(1-x)^{-2} \quad \cdot 1 = a_1$

$f''(x) = +2 \cdot 1 (1-x)^{-3} \cdot \quad = a_2$

$f'''(x) = 3 \cdot 2 \cdot 1 (1-x)^{-4}$

$\Rightarrow \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ geometric series

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Panel 7

Suppose $f(x) = e^x = \sum_{n=0}^{\infty} a_n x^n$. Find a_n

$f(0) = 1 = e^0$

$f'(0) = e^0 = 1$

$f''(0) = 1$

$f'''(0) = 1$

$a_n = \frac{f^{(n)}(c)}{n!}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$f'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

$= e^x$

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Panel 8

Ex: Suppose $e^{2x} = \sum_{n=0}^{\infty} a_n (x-1)^n$ $c=1$.

$f(0) = 1 = 2^0$

$f'(0) = 2 = 2^1$, $f'(x) = 2e^{2x}$

$f''(0) = 4 = 2^2$ $f''(x) = 4e^{2x}$

$\vdots$

$a_n = \frac{f^{(n)}(c)}{n!}$

$\Rightarrow a_n = \frac{2^n}{n!}$

$e^{2x} = \sum_{n=0}^{\infty} \frac{2^n}{n!} (x-2)^n$

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Panel 9

Ex: Find $\left.\frac{d^{11}}{dx^{11}}\right|_{x=0}$, $f(x)=\frac{x^3}{1-x^2}$

Too lazy to take 11-deriv!

$$\frac{x^3}{1-x^2}=x^3\frac{1}{1-x^2}=x^3\frac{1}{1-(x^2)}=x^3\sum_{n=0}^{\infty}(x^2)^n=\sum_{n=0}^{\infty}x^{2n+3}=$$

$$=1\cdot x^3+x^5+x^7+x^9+x^{11}+\dots$$

$a_3=1$ $\qquad f^{(2n)}(0)=0$

$a_5=1$

$a_7=1$

$a_{11}=1=\frac{f^{(11)}(0)}{11!} \rightarrow f^{(11)}(0)=11!$

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Panel 10

Suppose $\cos(x)=\sum_{n=0}^{\infty}a_n x^n$. Find a_n

$f(x)=\cos(x)$	$f(0)=1$	$a_0=\frac{1}{0!}$	$\cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$
$f'(x)=-\sin(x)$	$f'(0)=0$	$a_1=\frac{0}{1!}$	
$f''(x)=-\cos(x)$	$f''(0)=-1$	$a_2=\frac{-1}{2!}$	
$f'''(x)=\sin(x)$	$f^{(3)}(0)=0$	$a_3=0$	
$\vdots$		$a_4=\frac{1}{4!}$	

HW If $\sin(x)=\sum_{n=0}^{\infty}a_n x^n$, find the terms

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Panel 11

Ex: Suppose $e^{2x} = \sum_{n=0}^{\infty} a_n (x-1)^n$

Ex: Find $\frac{d^{n}}{dx^{n}}$ for $f(x) = \frac{x^3}{1-x^2}$

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Panel 12

Def: If f is a C^{∞}-function, then

$$T_f(x,c) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called Taylor Series for f (or if $c=0$, MacLaurin Series)

Ex: Take $f(x) = x^3 + 2x^2 + 3x + 4$. Find $T_f(x,0)$ and $T_f(x,1)$ and verify that $T_f(x,0) = T_f(x,1) = f(x)$

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Panel 13

$f(x) = x^3 + 2x^2 + 3x + 4$

$f(x) = x^3 + 2x^2 + 3x + 4 \qquad f(0) = 4 \qquad f(1) = 10$

$f'(x) = 3x^2 + 4x + 3 \qquad f'(0) = 3 \qquad f'(1) = 10$

$f''(x) = 6x + 4 \qquad f''(0) = 4 \qquad f''(1) = 10$

$f'''(x) = 6 \qquad f'''(0) = 6 \qquad f'''(1) = 6$

$f''''(x) = 0$

$\vdots$

$$T_f(x,0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 4 + \frac{3}{1!}x + \frac{4}{2!}x^2 + \frac{6}{3!}x^3 + 0$$

$$= 4 + 3x + 2x^2 + x^3$$

$$T_f(x,1) = 10 + \frac{10}{1!}(x-1) + \frac{10}{2!}(x-1)^2 + \frac{6}{3!}(x-1)^3 = 10 + 10(x-1) + 5(x-1)^2 + (x-1)^3$$

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Panel 14

Find the Taylor series centered at $x_0 = 0$ for:

e^x ✓

$\sin(x)$ ✓ HW

$\cos(x)$ ✓

$\ln(1-x)$ HW

Later

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Panel 15

Find the Taylor series centered at zero for:

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f^{(n)}(0) = ?$$

$$f(0) = 0$$

$$f'(x) = \begin{cases} \frac{2}{x^3} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h} =$$

$$= \lim_{h \to 0} \frac{1/h}{e^{1/h^2}} \underset{\frac{1}{h} = u}{=} \lim_{u \to \pm\infty} \frac{u}{e^{u^2}} \overset{\text{l'Hopital}}{=} \lim_{u \to \pm\infty} \frac{1}{2u e^{u^2}} = 0$$

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