Complex Analysis Exam 2

*This is a take-home exam. You may use the book or your notes as you wish, but you **must** complete each problem on your own. Show all your work (and be neat). Due: Friday, May 9, 2014 - **no** exceptions!*

1. Perform the following integrations along the indicated contours. You can use any method you like.

   a) $\int_C \frac{e^z}{z-2} dz$, C the unit circle $|z|=1$

   b) $\int_C \frac{e^{iz}}{z^3} dz$, C the square with corners 1, i, -1, and –i.

   c) $\int_C \frac{\cos(2z)}{z(z-2)} dz$, C the circle $|z-3|=2$

   d) $\int_C \frac{2z+1}{z(z+1)} dz$, C the circle $|z|=2$

2. If f is entire, $f(1) = 2$, f(0) = 0, and $|f'(z)| < M$ then show that $f(z) = 2z$

3. Find the Taylor series for each given function centered at the point $z_0 = 0$. Specify the radius of convergence for each series.

   a) $f(z) = z^3 \cos(z^2)$

   b) $f(z) = \frac{z}{3-2z}$

4. Find a Laurent series for the given function centered at the given point z_0 that converges in the specified domain and list, specifically, the residue.

   a) $f(z) = z^3 e^{\frac{1}{z}}$, $z_0 = 0$, convergent in domain including $z = 1$

   b) $f(z) = \frac{1}{3-4z+z^2}$, $z_0 = 0$, convergent in domain including $z = 2$

5. Consider the function $f(z) = \frac{e^z}{(3-z)(z-1)}$

   a) If you were to find a **Taylor** series for this function **centered at z = 0** converging for $|z| < R$, then R has what value?

   b) If you were to find the **Laurent** series **centered at** $z = i$ converging in the largest annulus $r < |z-i| < R$ including the point $z = 2$, then what are r and R?

6. Use the Residue Theorem to evaluate

   a) $\int_C z^4 e^{2/z} dz$, where C is the circle $|z-i| = 42$

   b) $\int_C \frac{e^z - 1}{(z-1)(z-2)(z-4)} dz$, where C is the circle $|z| = 3$

7. Recall that a function $f(z)$ has a **pole of order k at a point z_0** if the Laurent series centered at z_0 has only finitely many non-zero coefficients for $n < 0$ and $a_n = 0$ for all $n < -k$. For example(*), a function with a pole of order 2 at z_0 has a Laurent series

$$f(z) = \sum_{n=-2}^{\infty} a_n (z - z_0)^n = a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0)^2 + \ \ldots$$

with $a_{-2} \neq 0$, i.e. a pole of order 2 means that the smallest negative non-zero coefficient of the Laurent series is a_{-2}.

a) Prove the following **Theorem**: If $f(z)$ is analytic in a domain D except at a point $z = z_0$, and that point z_0 is a pole of order 1, then

$$Res(f, z_0) = a_{-1} = \lim_{z \to z_0} \ (z - z_0)f(z)$$

b) The function $f(z) = \frac{\sin(2z)}{z^2(z+1)(z-3)}$ has poles of order 1 at $z_0 = -1, 0, 3$. Use the above theorem to find the residues at those poles. Then use the Residue theorem to find

$$\int_C \frac{\sin(2z)}{z^2(z+1)(z-3)} dz$$

where C is a circle centered at zero with radius 2.

EXTRA CREDIT: Prove that if $f(z)$ has a pole of order 2 at z_0 then

$$Res(f, z_0) = \lim_{z \to z_0} \left(\frac{d}{dz} (z - z_0)^2 f(z) \right)$$

Hint: Write out the series for f centered at z_0 with a pole of order 2, multiply that series by $(z - z_0)^2$ and simplify. Then take the derivative and finally take the limit. If you do this carefully, the proof will do itself.

(*) Here are a few more examples to illustrate poles of order k:

- $g(z) = \frac{1}{z(z-1)}$ has poles of order 1 each at z = 0 and z = 1. To see that, we need to find the Laurent series centered at z = 0 and another one centered at z = 1.
  The Laurent series centered at 0 is:
  $$\frac{1}{z(z-1)} = \frac{1}{z} \frac{-1}{1-z} = \frac{1}{z}\left(-\sum_{n=0}^{\infty} z^n\right) = -z^{-1} - 1 - z - z^2 - z^3 - \cdots$$
  The Laurent series centered at 1 is:
  $$\frac{1}{z(z-1)} = \frac{1}{z-1} \frac{1}{z} = \frac{1}{z-1} \frac{1}{1+(z-1)} = \frac{1}{z-1} \left(\sum_{n=0}^{\infty} (-1)^n (z-1)^n\right) = (z-1)^{-1} - 1 + (z-1) \ldots$$
  Note that $Res(f, 0) = -1$ and $Res(f, 1) = 1$ - check that against the above theorem
- $f(z) = \frac{e^z - 1}{z^3}$ has a pole of order 2 (not 3 which you might originally think) at z = 0 because
  $$\frac{e^z - 1}{z^3} = \frac{1}{z^3}\left(\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) - 1\right) = \frac{1}{z^3}\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) = z^{-2} + \frac{z^{-1}}{2!} + \frac{1}{3!} + \frac{z}{4!} + \cdots$$
  Note that the residue of f is ½! – check this against the above extra credit problem.