Complex Practice Exam 1

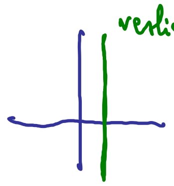
*This practice exam contains **sample** questions. The actual exam will have fewer questions, and may contain questions not listed here.*

1. Be prepared to explain the following concepts, definitions, or theorems:
   - A complex number, polar coordinates, rectangular coordinates
   - Add, Multiply, Sub, Div, Conjugate, abs Value - graphical interpretations of these
   - Complex roots, graphically and algebraically
   - The limit of a complex function f(z) as z approaches c is L
   - Continuity of a complex function f(z) at a point z = c
   - The complex derivative of a function f(z)
   - Analytic function and Entire function
   - CR equations
   - f(z) analytic & f'(z) = 0, f(z) analytic & f-conjugate analytic, f(z) analytic and |f(z)| constant
   - Harmonic function and harmonic conjugate of a function u (incl. how to find)
   - Arg(z), arg(z), e^z, sin(z), cos(z), log(z), and Log(z)
   - Euler's Formula, De Moivre's Formula
   - Complex parametric functions z(t), their integrals and derivatives
   - Different paths (line segments and circles)
   - Contour Integrals

2. Describe the set of points z such that

   (a) $\text{Re}(z) = 1$

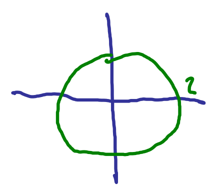
   If $z = x+iy$ then $\text{Re}(z) = x = 1 \Rightarrow$

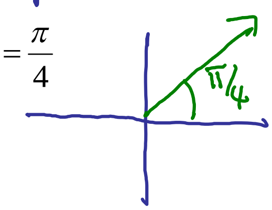
   vertical line

   (b) $|z-1| = 2$

   $\sqrt{(x-1)^2+y^2} = 2 \Rightarrow$ circle

   (c) $Arg(z) = \frac{\pi}{4}$

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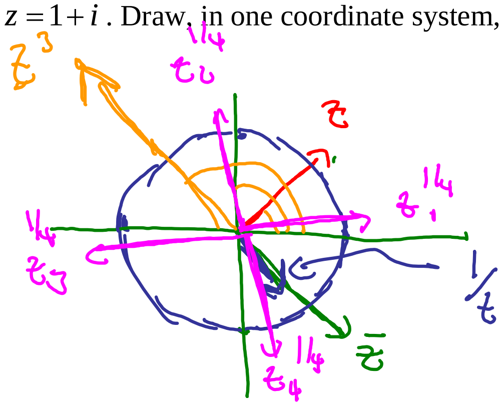
3. Let $z = 1+i$. Draw, in one coordinate system, $\bar{z}$, $\frac{1}{z}$, z^3, and $z^{\frac{1}{4}}$

   $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{|z|^2}\cdot\bar{z}$

4. Compute/simplify the following and find real and imag parts:

a) $\left|\overline{(1+i)}(1-i)i\right| = \left|\overline{(1+i)}\right| \cdot |1-i| \, |i| =$

$= \sqrt{2} \cdot \sqrt{2} \cdot 1 = 2$

(b) $\dfrac{i(1+i)^3}{(1-i)^2} = e^{i\pi/2} \cdot \left(\sqrt{2}\, e^{i\pi/4}\right)^3 \cdot \left(\sqrt{2}\, e^{-i\pi/4}\right)^{-2} = \sqrt{2}\, e^{i\left(\frac{\pi}{2} + \frac{3\pi}{4} - \frac{\pi}{2}\right)}$

$= \sqrt{2}\, e^{i\frac{3\pi}{4}} = -(1-i)$

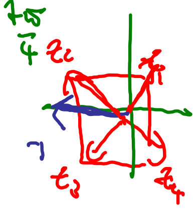
(c) $(1+i)^6 = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^6 = 2^3 e^{i\frac{6\pi}{4}} = -8i$

(d) $\dfrac{2+2i}{-\sqrt{3}+i} \cdot \dfrac{i+\sqrt{3}}{i+\sqrt{3}} = \dfrac{2i + 2\sqrt{3} - 2 + 2i\sqrt{3}}{-1-3} = \dfrac{2\sqrt{3}-2 + i(2+2\sqrt{3})}{-4}$

Re Im

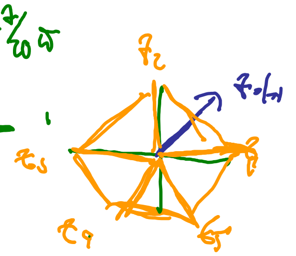
5. Find the fourth roots of -1, i.e. $\sqrt[4]{-1}$, and display them graphically. Do the same for the fifth roots of -1 and of (1+i).

$-1 = e^{i\pi} \Rightarrow \sqrt[4]{-1} = e^{i\left(\frac{\pi + 2k\pi}{4}\right)} = e^{i\pi/4}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$



$(1+i) = \sqrt{2}\, e^{i\pi/4} \Rightarrow$

$\sqrt[5]{1+i} = 2^{1/10} e^{i\frac{\pi/4 + 2k\pi}{5}} = 2^{1/10} e^{i\pi/20}, 2^{1/10} e^{i\frac{9}{20}\pi}, 2^{1/10} e^{i\frac{17}{20}\pi}, 2^{1/10} e^{i\frac{25}{20}\pi}, 2^{1/10} e^{i\frac{33}{20}\pi}$



6. Consider the following questions, involving limits and continuity of complex functions. Remember that limits can be taken in different directions, and for complicated limits there is l'Hospital's rule as long as the function is C-differentiable.

a) If $f(z) = \dfrac{x-iy}{x+iy}$, then f is clearly undefined at $z = 0$. Can you define *f(0)* in such a way that the new function is continuous at every point in the complex plane? *Hint: focus on the point (0,0).*

$$\lim_{(x,y)\to(0,0)} \frac{x-iy}{x+iy} = \begin{cases} 1 & \text{if } x \to 0,\ y = 0 \\ -1 & \text{if } y = 0,\ x \to 0 \end{cases} \quad \nexists \lim$$

The limit does not exist so no definition of f(0) would make the function continuous

b) Say $f(z)=\dfrac{z^9+z-2i}{z^{15}+i}$ Can you define *f(i)* in such a way that the new function is continuous at every point in the complex plane?

$$\lim_{z\to i}\frac{z^9+z-2i}{z^{15}+i}=\left(\frac{0}{0}\right)=\lim_{z\to i}\frac{9z^8+1}{15z^{14}}=\frac{10}{-15}=-\frac{2}{3}$$

Thus $f(z)=\begin{cases}\dfrac{z^9+z-2i}{z^{15}+i}, & z\neq i\\[2mm] -\frac{2}{3}, & z=i\end{cases}$ is continuous for all z.

c) Find $\displaystyle\lim_{z\to 1}\frac{1+z^6}{1+z^{10}}$, $=\left(\frac{0}{0}\right)=\displaystyle\lim_{z\to 1}\frac{6z^5}{10z^9}=\frac{6}{10}=\frac{3}{5}$

d) $\displaystyle\lim_{z\to i}\frac{1+z^6}{1-z^{10}}=\frac{0}{2}=0$

e) $\displaystyle\lim_{z\to i}\frac{1+z^6}{1+z^{10}}$ = see above

7. Consider the following questions about analytic functions.

a) If $f(z)=\dfrac{1}{\left(z^2+1\right)^2}$ then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of *f*.

f is analytic $\forall z\neq \pm 1$. Moreover:

$$f'(z)=-2(z^2+1)^{-3}\cdot 2z=\frac{4z}{(z^2+1)^3}$$

b) If $f(z)=\underbrace{x^3-3xy^2}_{u}+i\underbrace{\left(3x^2y-y^3\right)}_{v}$ then determine where, if at all, the function is analytic. If it is analytic, find the complex derivative of *f*.

$u_x=3x^2-3y^2$ $\quad v_y=3x^2-3y^2$ $\quad\Rightarrow\quad u_x=v_y$ and

$u_y=-6xy$ $\quad v_x=6xy$ $\quad u_y=-v_x$

$\Rightarrow$ f is entire. $\quad f'=u_x+iv_x=\underline{3x^2-3y^2+i6xy}$

Note: $f'=3(x^2-y^2+2ixy)=3z^2$ so that $f(z)=z^3$

8. Decide which of the following functions are analytic, and in which domain they are analytic. If a function is analytic, find its complex derivative:

(a) $f(z) = \dfrac{e^z + 1}{e^z - 1}$

Analytic for all $e^z \neq 1$

$\Leftrightarrow \forall z \neq 2k\pi i, k \in \mathbb{Z}$

(b) $f(z) = x^3 + 3ix^2 y - 3xy^2 + x - iy^3 + iy$

very similar to $f(z)$

9. Consider the function $u(x, y) = e^x \sin(y)$. Is it harmonic ? If so, find its harmonic conjugate.

$u_x = e^x \sin(y)$, $u_{xx} = e^x \sin(y)$

$u_y = e^x \cos(y)$, $u_{yy} = -e^x \sin(y)$

$\Rightarrow u$ is harmonic

$v_y = u_x = e^x \sin(y)$

$\Rightarrow v = \int e^x \sin(y)\,dy =$

$= -e^x \cos(y) + c(x)$

$v_x = -e^x \cos(y) + c'(x) = -u_y$

$\Rightarrow c' = 0 \Rightarrow v = -e^x \cos(y)$

Do the same for (a) $u(x, y) = x^3 - 2xy + xy^3$

$u_x = 3x^2 - 2y$, $u_{xx} = 6x$

$u_y = x + 3x^2 y$, $u_{yy} = 3x^2$ so not harmonic

And for $u(x, y) = e^y \cos(x)$

$u_x = -e^y \sin(x)$, $u_{xx} = -e^y \cos(x)$ $\Rightarrow u$ is harmonic

$u_y = e^y \cos(x)$, $u_{yy} = e^y \cos(x)$

$v_y = u_x = -e^y \sin(x)$ $\Rightarrow v = -\int e^y \sin(x)\,dy = -e^y \sin(x) + C(x)$

$\Rightarrow v(x,y) = -e^y \sin(x) + c$

10. Please find the following numerical answers:

(a) e^{2+2i} $= e^2 e^{2i} = e^2(\cos(2) + i\sin(2)) = \cos(2)e^2 + i\sin(2)e^2$

(b) $\cos(\pi+i)$ $= \frac{1}{2}\left(e^{i(\pi+i)} + e^{-i(\pi+i)}\right) = \frac{1}{2}\left(e^{i\pi}e^{-1} + e^{-i\pi}e^{1}\right) =$

$= \frac{1}{2}\left(-e^{-1} - e^{1}\right) = -\frac{1}{2}\left(e + \frac{1}{e}\right)$

$= \underline{-\cosh(1)}$

(c) $\sin\left(i - \frac{\pi}{2}\right)$

$= \frac{1}{2i}\left(e^{i(i-\pi/2)} - e^{-i(i-\pi/2)}\right) = \frac{1}{2\cancel{i}}\left(e^{-1}(-\cancel{i}) - e^{1}(\cancel{i})\right) = \underline{-\cosh(1)}$

(d) $\log(-2)$ $= \log\left(2e^{i\pi}\right) = \underline{\ln(2) + i(\pi + 2k\pi)}$

(e) $Log(1+i)$ $= Log\left(\sqrt{2}e^{i\frac{\pi}{4}}\right) = \underline{\ln(\sqrt{2}) + i\frac{\pi}{4}}$

11. Solve the following equations for z.

(a) $z^4 + 1 = 0$ $\Rightarrow z = (-1)^{1/4} \Rightarrow z = \underline{e^{i\pi/4},\ e^{i3\pi/4},\ e^{i5\pi/4},\ e^{i7\pi/4}}$

(b) $\left|e^{2z}\right| = 3$

$\left|e^{2z}\right| = \left|e^{2(x+iy)}\right| = e^{2x}\left|e^{2iy}\right| = e^{2x} = 3 \Rightarrow \underline{x = \ln(3)}$, $\underline{\text{any } y}$

(c) $\sin(z) = 3i$

$\frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) = \frac{1}{2i}\left(e^{ix}e^{-y} - e^{-ix}e^{y}\right) = \frac{1}{2i}\left[(\cos(x) + i\sin(x))e^{-y} - (\cos(-x) + i\sin(-x))e^{y}\right]$

(d) $e^{4z} = 1$

easy

$= \frac{1}{2i}\left(\cos(x)\left(e^{-y} - e^{y}\right) + i\sin(x)\left(e^{-y} + e^{y}\right)\right) = 3i$

$\Rightarrow i\cos(x)\left(e^{y} - e^{-y}\right) + \sin(x)\left(e^{-y} + e^{y}\right) = 6i$

Too much!

(e) $\cos(z) = i\sin(z)$

hard

$\Rightarrow \underline{x = 0 \text{ or } \pi}$ and $e^{y} - e^{-y} = 6i$

$\Rightarrow \frac{1}{2}\left(e^{y} - e^{-y}\right) = 3 \Rightarrow \underline{y = \sinh^{-1}(3)}$

12. Use the **definition** of derivative to show that the functions $f(z) = \text{Re}(z)$ is nowhere differentiable.

$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{(x,y)\to(x_0,y_0)} \frac{x - x_0}{(x-x_0) + i(y-y_0)}$

$x = x_0$, lim is 0
$y = y_0$, lim is 1

$\Big\}$ limit never exists $\Rightarrow f(z)$ is nowhere diff'ble

Use the CR equations to show that the function $f(z) = \bar{z}$ is nowhere differentiable.

$f(z) = \bar{z} = x + i(-y)$

$u = x \qquad u_x = 1, \ v_y = -1$ so not diff'ble

$v = -y$

Show that if v is the harmonic conjugate of u, then the product u v is harmonic.

Know: $u + iv$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

$(u \cdot v)_{xx} + (u \cdot v)_{yy} = 0$ as you can see when you expand this cleverly!

Prove that if $h(x, y)$ is a harmonic everywhere then the complex function $f(z) = \frac{\partial}{\partial x} h(x, y) - i \frac{\partial}{\partial y} h(x, y)$ is an analytic function for all z.

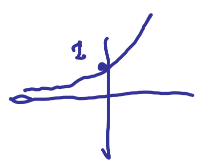
$f = h_x - i h_y$

$\Rightarrow u = h_x \quad \Rightarrow u_x = h_{xx}, \ v_y = -h_{yy}$ but $h_{xx} = -h_{yy}$ ✓

$v = -h_y$ also $u_y = h_{xy} = h_{yx} = -v_x$ so CR checks out

15 Show that $|e^z| \le 1$ if $\mathrm{Re}(z) \le 0$

$|e^z| = e^x \le 1 \quad \Rightarrow x \le 0$



$\Leftrightarrow \mathrm{Re}(z) \le 0$

16 State De Moivre's formula. Then use it to prove the trig identity $\sin(2x) = 2\sin(x)\cos(x)$

$(e^{it})^2 = e^{i2t} \Leftrightarrow (\cos(t) + i\sin(t))^2 = \cos(2t) + i\sin(2t)$

$\Rightarrow \cos^2(t) - \sin^2(t) + 2i\cos(t)\sin(t) = \cos(2t) + i\sin(2t)$

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17 Show that the function e^{iz} is periodic with period 2π

$e^{i(z+2\pi)} = e^{iz+2\pi i} = e^{iz} e^{2\pi i} = e^{iz} \cdot 1$ #

18 Show that the function *sin(z)* is unbounded

19 Show that the function $f(z) = z\bar{z} + z + \bar{z} + 2x$ cannot be an analytic function.

$f(z) = x^2 + y^2 + 2x + 2x =$

$= x^2 + y^2 + 4x$ always real

$\Rightarrow$ can't be analytic!

20 Prove that $\sin^2(z) + \cos^2(z) = 1$ (Hint: there are several ways to do this. One slick way involves taking the derivative of $f(z) = \sin^2(z) + \cos^2(z)$. Another possibility is to work with the actual definitions of sin and cos)

$\sin^2(z) + \cos^2(z) = \left(\frac{1}{2i}(e^{iz} - e^{-iz})\right)^2 + \left(\frac{1}{2}(e^{iz} + e^{-iz})\right)^2 = \ldots = 1$

FOIL IT OUT!

21 Prove the following theorem: If *f(z)* is an analytic function with values that are always imaginary, then the function must be constant.

use CR equations with $u(x,y) = 0 \;\forall (x,y)$
to show that $v_x = v_y = 0 \Rightarrow v = \text{constant}$

22 Find complex parametric functions representing the following paths:

(a) a straight line from –i to i

$\underline{z(t) = -i + 2it}$, $t \in [0,1]$

(b) the right half of a circle from –i to i,

$\underline{z(t) = e^{it}}$, $t \in [-\pi/2, \pi/2]$

(c) a straight line from -1 – 2i to 3 + 2i

$z(t) = -1 - 2i + (3 + 2i - (-1 - 2i))t$, $t \in [0,1]$

line from A to B: $z(t) = A + (B - A) \cdot t$, $t \in [0,1]$

(d) a circle centered at 1+i of radius $|2-3i| = \sqrt{13}$

$\sqrt{13}e^{it}+(1+i)$, $t\in[0,2\pi]$

23 Evaluate

a. z'(t) for $z(t)=\cos(2t)+i\sin(2t)$

$z'(t) = -2\sin(2t) + 2i\cos(2t)$

b. $\int_0^{\pi} z(t)dt$ for $z(t)=(5+4i)e^{3it}$

$\int_0^{\pi}(5+4i)e^{3it}dt = (5+4i)\int_0^{\pi}e^{3it}dt = (5+4i)\cdot\frac{1}{3}(e^{3\pi}-1)$

24 Evaluate

a. $\int_\gamma iz^2+3dz$ where γ is a line segment from -1-i to 1+i

$z(t) = -1-i+(1+i-(-1-i))t = -1-i+(2+2i)t$

$\Rightarrow \int_\gamma iz^2+3\,dz = \int_0^1\left(i[-1-i+(2+2i)t]^2+3\right)(2+2i)dt =$

Wolfram Alpha

b. $\int_\gamma \frac{1}{\bar{z}}dz$ where γ is a circle radius 2 centered at the origin

$z(t) = 2e^{it}$, $t\in[0,2\pi]$

$\int_\gamma \frac{1}{\bar{z}}dz = \int_0^{2\pi}\frac{1}{\cancel{2}e^{-it}}\cancel{2}ie^{it}dt = i\int_0^{2\pi}e^{it}\cdot e^{it}dt = i\int_0^{2\pi}e^{2it}dt = 0$

c. $\int_\gamma \frac{1}{z^3}dz$ where γ is a circle radius 2 centered at the origin

$\int_\gamma \frac{1}{(2e^{it})^3}i2e^{it}dt = \frac{2i}{8}\int_0^{2\pi}e^{-3it}e^{it}dt = \frac{i}{4}\int_0^{2\pi}e^{-2it}dt = 0$

25 Is 1 raised to any power (integer or otherwise) always equal to 1?

yes