Panel 1

Complex - The final lecture

Laurent Thm: If f is analytic in $R_1 < |z-z_0| < R_2$

then $f(z) = \sum_{-\infty}^{\infty} a_n (z-z_0)^n$

Residue

with $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

$a_2 = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^3}\, dz$ $\quad f(z_0) = a_0 = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z-z_0}\, dz$

$a_{-2} = \frac{1}{2\pi i}\int_{C_R} f(z)(z-z_0)$ $\quad a_{-1} = \frac{1}{2\pi i}\int_{C_R} f(z)\, dz$

Panel 2

Residue Thm (Simple Version)

If f is analytic in $0 < |z-z_0| < R$ then

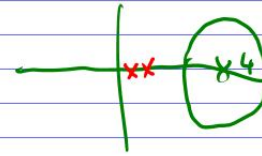
$$\int_C f(z)\, dz = 2\pi i\, a_{-1} = 2\pi i\, \text{Res}(f, z_0)$$

Ex $\int_{|z|=1} \frac{\sin(z)}{z^4}\, dz$; $\boxed{\frac{\sin(z)}{z^4} = z^{-3} - \frac{z^{-1}}{3!} + \frac{z}{5!} - \cdots,\ a_{-1} = -\frac{1}{6}}$

$= 2\pi i\left(-\frac{1}{6}\right)$

Panel 3

Ex: $\int_C \frac{-2}{z(z-1)}\,dz$ a) $|z-4|=1$



b) $|z|=1/2$

singularities at $z=0$, $z=1$

c) $|z|=2$

$$\int_{|z|=1/2} \frac{-2}{z(z-1)}\,dz = \int \frac{f(z)}{z}\,dz = 2\pi i\, f(0) = 2\pi i \frac{-2}{-1} =$$

$f(z)$

$$\mathrm{Res}\left(\frac{-2}{z(z-1)}, 0\right) = +\frac{2}{z}\,\frac{1}{1-z} = \frac{2}{z}\sum_{n=0}^{\infty} z^n =$$

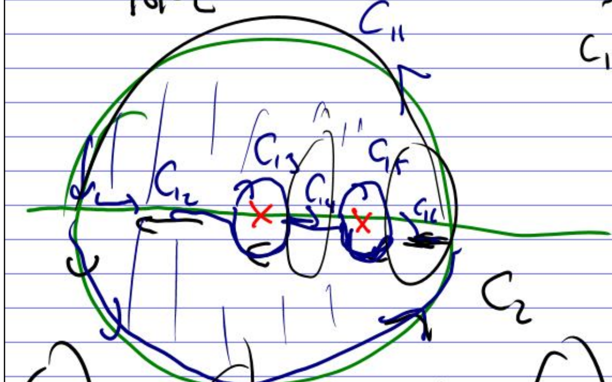
$$= \frac{2}{z}\left(1+z+z^2+\dots\right) = \frac{2}{z} + 2 + 2z + \dots$$

a_{-1}

Panel 4

(c) $\int_{|z|=2} \frac{-2}{z(z-1)}\,dz$

$$\int_{C_1} f = \int_{C_2} f = 0$$



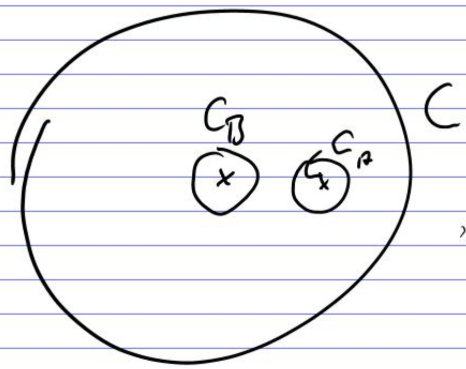
$$\int_{C_1} + \int_{C_2} = 0$$

$$\int_{C_{11}} + \cancel{\int_{C_{12}}} + \int_{C_{13}} + \cancel{\int_{C_{14}}} + \cancel{\int_{C_{15}}} + \int_{C_{16}} + \int_{C_{21}} + \cancel{\int_{C_{22}}} + \cancel{\int_{C_{23}}} + \int_{C_{24}} + \cancel{\int_{C_{25}}} + \int_{C_{26}} =$$

$$\int_C - \left(\int_{C_A} + \int_{C_B}\right) = 0$$

Panel 5

$$\int_C f(z)\,dz = \int_{C_A} f(z)\,dz + \int_{C_B} f(z)\,dz = 2\pi i\left(\mathrm{Res}(f,0) + \mathrm{Res}(f,1)\right)$$



Thus

$$\int_{|z|=2} \frac{-2}{z(z-1)}\,dz = 2\pi i\cdot\left(\mathrm{Res}(f,0) + \mathrm{Res}(f,1)\right)$$

Panel 6

Thus: $$\int_{|z|=2} \frac{-2}{z(z-1)}\,dz = \int_{|z|=\varepsilon} \frac{-2}{z(z-1)}\,dz + \int_{|z-1|=\varepsilon} \frac{-2}{z(z-1)}\,dz$$

$= 0$

$$2\pi i\left(\underbrace{\mathrm{Res}(f,0)}_{2} + \underbrace{\mathrm{Res}(f,1)}_{-2}\right)$$

$$\mathrm{Res}(f,1) = \frac{-2}{z(z-1)} = \frac{-2}{z-1}\,\frac{1}{z} = -\frac{2}{z-1}\,\frac{1}{1+z-1} =$$

$$= -\frac{2}{z-1}\,\frac{1}{1-(-(z-1))} = -\frac{2}{z-1}\sum_{n=0}^{\infty}\left(-(z-1)\right)^n$$

$$= -\frac{2}{z-1}\sum_{n=0}^{\infty}(-1)^n(z-1)^n = -\frac{2}{z-1}\left(1-(z-1)+(z-1)^2-\dots\right.$$

$\Rightarrow \mathrm{Res}(f,1) = -2$

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Panel 8

Thus: $\displaystyle\int_{|z|=1} \frac{-2}{z(z-1)}\,dz = \text{Res}(f, z=0) + \text{Res}(f, z=1) =$

$= 2\pi i(2) + (2\pi i)(-2) = \underline{0}$

Residue Thm: Suppose f is analytic in D except at finitely many points $z_1, z_2, \ldots, z_n$. Let C be a curve in D around z_j, $j=1,\ldots,n$.

Then
$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j)$$

Panel 9

Residue Thm is the culmination of an UG course in Complex.

Panel 10

Find the residues for the functions as given:

a) $\mathrm{Res}(f,0) = \frac{1}{6!}$, $f(z) = z^5 \cos(1/z) = z^5\left(1 - \frac{(\frac{1}{z})^3}{2!} + \frac{(\frac{1}{z})^4}{4!} - \right.$

$= z^5 - \frac{z^3}{2!} + \frac{z}{4!} - \frac{1}{z\,6!}$

b) $\mathrm{Res}(f,0) = 1$, $f(z) = \frac{1}{z^2} e^{z}$

$\frac{1}{z^2}\left(1 + z + \frac{z^2}{1!} + \frac{z^3}{3!} + \dots\right)$

$z^{-2} + z^{-1} + \frac{1}{2!} + \frac{z}{3!}$

c) $\mathrm{Res}(f,0)$, $f(z) = \frac{3}{z(z+2)}$

d) $\mathrm{Res}(f,-2)$, $f(z) = \frac{3}{z(z+2)}$

e) $\mathrm{Res}(f,3)$, $f(z) = \frac{3}{z(z+2)}$

f) $\mathrm{Res}(f,0)$, $f(z) = \frac{1}{z^3(z-2)}$

Panel 11

$$f(z) = \frac{3}{z(z+2)}$$

$\mathrm{Res}(f,0)$: $\quad \frac{3}{z} \Big/ \frac{1}{z+2} = \frac{3}{z} \frac{1}{2\left(1+\frac{z}{2}\right)} = \frac{3}{z}\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n$

$3/2$ $\quad = \frac{3}{\cancel{z}\,2}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{2^n} = \frac{3}{2z}\left(1 - \frac{z}{2} + \dots\right)$

$\mathrm{Res}(f,-2)$: $\quad \frac{3}{z(z+2)} = \frac{3}{z+2}\cdot\frac{1}{z} = \frac{3}{z+2}\cdot\frac{1}{-2+(z+2)} =$

$$= \frac{3}{z+2}\left(-\frac{1}{2}\right)\frac{1}{1-(z+2)} = -\frac{3}{2(z+2)}\left(\sum(-1)^n(z+2)^n\right)$$

$\mathrm{Res}(f,3) = 0$ Radius of conv. of Taylor series

$-3/2$ centered at 3 is $R = 3$

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$\mathrm{Res}(f,0) = -\frac{1}{8}$ $\quad f(z) = \frac{1}{z^3(z-2)} =$

$$= \frac{1}{z^3}\,\frac{1}{z-2} = \frac{1}{z^3}\left(-\frac{1}{2}\right)\left(\frac{1}{1-z/2}\right)$$

$$= -\frac{1}{2z^3}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n =$$

$$= -\frac{1}{2z^3}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \frac{z^4}{2^4} + \dots\right)$$

$$= \quad -\frac{1}{8z} + \dots$$

Panel 13

$\underline{\underline{Ex:}}$ $\int_C \frac{5z-2}{z(z-1)}\,dz$, C is circle, radius ~~2~~ 0.5

$R(C,0)$ $= 2\pi i\left(\text{Res}(f,0) + \text{Res}(f,1)\right)$

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z}\cdot\frac{1}{z-1} = \left(5-\frac{2}{z}\right)\left(-\frac{1}{1-z}\right) =$$

$$= \left(5-\frac{2}{z}\right)\left(-\sum_{n=0}^{\infty} z^n\right) =$$

$$= \left(5-\frac{2}{z}\right)\left(-1-z-z^2-z^3-\dots\right)$$

$\text{Res}(f,0) = \underline{\underline{2}}$

Panel 14

$\frac{5z-2}{z(z-1)}$: $\text{Res}(f,1) = \underline{\underline{3}}$

$$\frac{5z-2}{z-1}\,\frac{1}{z} = \frac{5z-2}{z-1}\cdot\frac{1}{1+z-1} =$$

$$\frac{5z-2}{z-1}\sum(-1)^n(z-1)^n$$

$$\frac{5(z-1)+3}{(z-1)}\sum(-1)^n(z-1)^n$$

$$\left(5+\frac{3}{(z-1)}\right)\left(1 - (z-1) + (z-1)^2 \dots\right)$$

Panel 15

Finding Residues

Consider $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0) =$

$$= \dots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

Def: If $a_n = 0 \ \forall n<0$: } Removable —

If $a_{-1}, a_{-2}, \dots a_{-k} \neq 0$, $a_n = 0 \ \forall n < -k$ } Pole of order n —

If $a_n \neq 0$ for inf. many $-k$ } Essential Singularity

Panel 16

Ex: $f(z) = \frac{z^2 - 2z + 3}{z-2}$ Pole, order 1

$g(z) = \frac{1}{z^2(z+1)}$ Pole at 0: $n=2$; Pole at -1: $n=1$

$h(z) = \frac{\sin(z)}{z^4}$ Pole at 0 $n = \cancel{4}\ 3$

$f(z) = \frac{z^2-9}{z-3}$ Removable

$g(z) = \frac{1-\cos(z)}{z^2}$ Removable

$h(z) = e^{1/z}$ Essential