Panel 1

Last Time:

Power series: $\sum a_n (z-z_0)^n$ for $|z-z_0|<R$

radius of conv

Taylor's Thm: f is analytic in $D_{z_0}(R)$ then

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \qquad |z-z_0|<R$$

and $$a_n=\frac{f^{(n)}(z_0)}{n!}$$

Proof: $z_0=0$, $\Rightarrow$ $f(\overset{z}{\cancel{z_0}})=\frac{1}{2\pi i}\int_{C_R}\frac{f(\overset{w}{\cancel{z}})}{\underset{w}{\cancel{z}}-\underset{z}{\cancel{z_0}}}\,d\overset{w}{\cancel{z}}$ Cauchy Int

Panel 2

$$\Rightarrow f(z)=\frac{1}{2\pi i}\int_{C_R}\frac{f(w)}{w-z}\,dw$$

$f^{(n)}(z)=\frac{n!}{2\pi i}\int\frac{f(z)}{(z-z_0)^{n+1}}dz$

$$=\frac{1}{2\pi i}\int\frac{f(w)}{w\left(1-\frac{z}{w}\right)}\,dw=\frac{1}{2\pi i}\int\frac{f(w)}{w}\,\frac{1}{1-\left(\frac{z}{w}\right)}\,dw$$

$$=\frac{1}{2\pi i}\int\frac{f(w)}{w}\cdot\sum\left(\frac{z}{w}\right)^n dw=$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int\frac{f(w)}{w^{n+1}}\cdot z^n\,dw=$$

$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(\int\frac{f(w)}{w^{n+1}}\,dw\right)z^n \qquad a_n=\frac{f^{(n)}(0)}{n!}$$

$$=\frac{1}{\cancel{2\pi i}}\sum_{n=0}^{\infty}\frac{\cancel{2\pi i}}{n!}f^{(n)}(0)\,z^n=\sum a_n z^n \quad \#$$

Panel 3

Special Power Series every educated person must know:

$$f(z) = \frac{1}{1-z} = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n$$

$$f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$f(z) = \cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

$$f(z) = \sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Panel 4

Find McLaurin Series for $f(z) = 1/_{1-z}$, $g(z) = z/_{z^2+4}$

$\overset{\shortparallel}{\text{Taylor}}$ series with center $z_0 = 0$

$$\frac{1}{1-z} = \sum_0^{\infty} z^n$$

$$g(z) = \frac{z^2}{z^2+4} = z^2 \frac{1}{4+z^2} = z^2 \frac{1}{4\left(1+\left(\frac{z}{2}\right)^2\right)} =$$

$$= \frac{z^2}{4} \cdot \frac{1}{1-\underbrace{\left(-\left(\frac{z}{2}\right)^2\right)}} = \frac{z^2}{4} \sum_{n=0}^{\infty} \left(-\left(\frac{z}{2}\right)^2\right)^n =$$

$$= \frac{z^2}{4} \sum (-1)^n \frac{z^{2n}}{4^n} = \sum_0^{\infty} (-1)^n \frac{z^{2n+2}}{4^{n+1}}$$

Panel 5

Find McLaurin Series for $f(z) = z e^{z^2}$

$$z \cdot e^{(z^2)} = z \cdot \sum_{n=0}^{\infty} \frac{(z^2)^n}{n!} = z \sum_{0}^{\infty} \frac{z^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{n!}$$

Find $\lim_{z \to 0} \frac{\sin(z)}{z}$ = and $\lim_{z \to 0} \frac{\cos(z)-1}{z} = 0$

$$\frac{1}{z}\left| z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right| = 1 \qquad 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots - 1$$

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \frac{z^6}{7!} + \dots \qquad \frac{1}{z}\left(-\frac{z^2}{2!} + \frac{z^4}{4!} + \dots \right) = -\frac{z}{2!} + \frac{z^3}{4!}$$

Find (approx) value of $\int_0^1 e^{-x^2} dx$ no

Panel 6

Find McLaurin Series for $f(z) = z e^{z^2}$

Find $\lim_{z \to 0} \frac{\sin(z)}{z}$ and $\lim_{z \to 0} \frac{\cos(z)-1}{z}$

Find (approx) value of $\int_0^1 e^{-x^2} dx = \int_0^1 1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} - dt$

$$= 1 - \frac{1}{3(1!)} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \dots$$

Panel 7

Find power series for $f(z) = 1/z$ centered at $\cancel{c=0}$ $c = 2$

$$\frac{1}{z} = \frac{1}{2 + (z-2)} = \sum_{0}^{\infty} ?\,(z-2)^n$$

$$= \frac{1}{2}\,\frac{1}{1 - \frac{-(z-2)}{2}} = \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{(z-2)}{2}\right)^n =$$

$$\frac{1}{1-z} = \sum z^n \qquad = \frac{1}{2}\sum (-1)^n \frac{(z-2)^n}{2^n} = \sum (-1)^n \frac{(z-2)^n}{2^{n+1}}$$

Find power series of e^z centered at $z = 1$:

$$e^z = \sum_{0}^{\infty} \frac{z^n}{n!}, \quad e^{(z-1)} = e^{z}e^{-1} \quad e = e^z$$

Panel 8

$$e^z = e\,e^{z-1} = \sum_{n=0}^{\infty} e\,\frac{(z-1)^n}{n!}$$

Panel 9

Radius of Convergence Theorem:

① If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is a power series, then it converges for $|z-z_0| < R$ where $R = \lim_{n\to\infty} \frac{a_{n+1}}{a_{n}}$

② If $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ is a Taylor series, then it converges for $|z-z_0| < R$ where R is the largest radius in which f is analytic

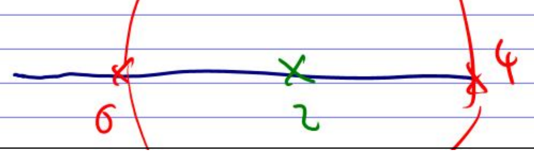
Panel 10

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-2)^n}{2^{n+1}} = \sum_{n=0}^{\infty} \underbrace{(-1)^n \frac{1}{2^{n+1}}}_{a_n} (z-2)^n$$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right| =$$

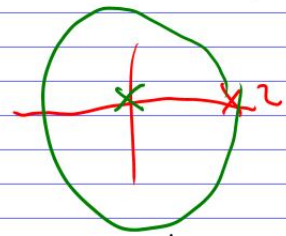
$$R = \lim \left| \frac{(-1)^{n+1}\ 1/2^{n+1}}{(-1)^n\ 1/2^n} \right| = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} = 2$$

$\sum (-1)^n \frac{(z-2)^n}{2^{n+1}}$ conv $|z-2| < 2$



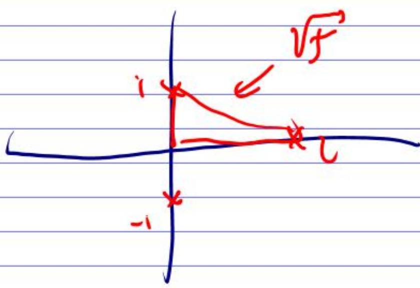
Panel 11

Ex: If $\frac{1}{2-z} = \sum_{n=0}^{\infty} a_n z^n$ for $|z|<R$, find $R = 2$



$R = 2$

Ex: If $\frac{1}{1+z^2} = \sum_{n=0}^{\infty} a_n (z-2)^n$ for $|z-2|<R$, find $R = \sqrt{5}$



Panel 12

What if f is analytic for $R_1 < |z-z_0| < R_2$

No power series for f!

Laurent Series

If f is analytic for $R_1 < |z-z_0| < R_2$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

Panel 13

Ex $f(z) = \frac{1}{2-z}$ Find series centered at $z_0 = 0$

a) for $|z| < 2$

$$\frac{1}{2-z} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

b) for $|z| > 2$

$$\frac{1}{2-z} = \frac{1}{z\left(\frac{2}{z}-1\right)} = \frac{1}{-z\left(1-\frac{2}{z}\right)} = -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$