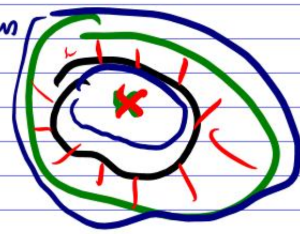
Panel 1

Foundational Theorems of Complex Analysis

1. Cauchy-Riemann Equations
2. Cauchy-Goursat Thm
3. Deformation thm
4. Path Independence thm
5. Cauchy Integral Formula
6. General Cauchy Int. Formula
7. Morera's thm
8. Cauchy's Inequality
9. Liouville's thm
10. Fund. thm. of Algebra



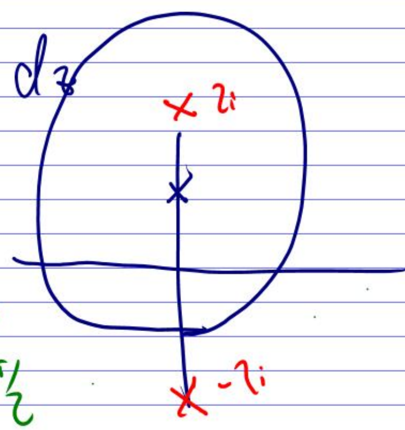
$$\int_C f\,dz = F(b) - F(a)$$

Panel 2

Let C be the circle $|z-i| = 2$. Find

$$\int_C \frac{1}{z^2+4}\,dz = \int \frac{1}{(z+2i)(z-2i)}\,dz \qquad f(z)$$

$z = -2i,\ 2i$



$$= \int_C \frac{f(z)}{z-2i}\,dz = 2\pi i\, f(2i) = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

b) $$\int_C \frac{1}{(z^2+4)^2}\,dz =$$

$$\int_C \frac{1}{(z+2i)^2 (z-2i)^2}\,dz = \int \frac{f(z)}{(z-2i)^2}\,dz = \frac{2\pi i}{1!} f'(2i)$$

Panel 3

$f(z) = \frac{1}{(z+2i)^2}, \quad f'(z) = -2(z+2i)^{-3}$

$\int \frac{f(z)}{(z-2i)^2}\,dz = \frac{2\pi i}{1!} f'(2i) = \frac{2\pi i}{1}(-2)(4i)^{-3}$

$\int_C \frac{1}{(z^2+4)}\,dz = \qquad |z-i| = 501$

$\int_C \frac{1}{(z+2i)(z-2i)}\,dz = \int \frac{A}{z+2i} + \frac{B}{z-2i}\,dz$

PFD

Panel 4

Thm: The Geometric Series $\sum_{n=0}^{\infty} z^n$ converges if $|z|<1$ and diverges if $|z|>1$.

$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \dots + z^n + \dots$

Ex $z = \frac{1}{2}$: $\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots$

$S_N = 2\left(1 - \left(\frac{1}{2}\right)^{N+1}\right)$

$S_N = \sum_{n=0}^{N}\left(\frac{1}{2}\right)^n = 1 + \cancel{\left(\frac{1}{2}\right)} + \cancel{\left(\frac{1}{2}\right)^2} + \dots \cancel{\left(\frac{1}{2}\right)^N}$

$-\ \frac{1}{2}S_N = \cancel{\frac{1}{2}} + \cancel{\left(\frac{1}{2}\right)^2} + \cancel{\left(\frac{1}{2}\right)^3} + \dots \left(\frac{1}{2}\right)^{N+1}$

$\frac{1}{2}S_N = 1 - \left(\frac{1}{2}\right)^{N+1}$

Thus:

$\lim_{N\to\infty} S_N = 2(1-0) = \underline{\underline{2}}$

$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = 2$

Panel 5

Panel 6

$1 + 0.5 + 0.25 + 0.125 + \quad =$

Similarly: $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ Prove as HW

Ex $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \frac{1}{1-\frac{3}{4}} = 4$

$\sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{1}{1-\frac{9}{10}} = 10$

$\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n = \infty$

$\sum_{n=3}^{\infty} z^n = z^3 + z^4 + z^5 + \ldots = z^3 \left(\underline{1 + z + z^2 + \ldots}\right) \Rightarrow \frac{z^3}{1-z}$

Panel 7

Def. A series $\sum_{n=0}^{\infty} z_n$ converges if sequence of partial sums $S_N = \sum_{n=0}^{N} z_n$ converges

Def. A series $\sum_{n=0}^{\infty} z_n$ converges absolutely if $\sum_{n=0}^{\infty} |z_n|$ converges

Thm: If $\sum_{n=0}^{\infty} z_n$ converges absolutely then $\sum z_n$ conv.

Thm: If $\sum_{n=0}^{\infty} z_n$ converges $= L$, then $\lim_{n\to\infty} z_n = 0$ (Divergence Test)

Proof: $S_N = z_0 + z_1 + z_2 + \dots + z_N$

$$\lim_{N\to\infty} S_N - S_{N-1} = L - L = 0 = \lim_{N\to\infty} z_N$$

Panel 8

Def: A series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n =$$

is called a Power Series centered at z_0

Ex: $\sum_{n=0}^{\infty} z^n$ is a power series centered at 0 with $a_n = 1$

$a_0 + a_1(z - z_0) + a_2(z-z_0)^2 + $

Panel 9

<u>Taylor's Theorem:</u> If f is analytic in the disk $|z-z_0|<R$ then

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \forall\ |z-z_0|<R$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

In other words: every analytic function is a sum of monomials !!!

Panel 10

<u>Ex:</u> Find Taylor Series centered at zero for:

$$f(z)=e^z,\ g(z)=z^3e^{2z},\ h(z)=\sin(z),\ k(z)=\cos(z)$$

$$e^z=\sum_{n=0}^{\infty} a_n z^n \quad ,\ a_n=\frac{f^{(n)}(0)}{n!}=\frac{e^0}{n!}=\frac{1}{n!}$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{n!}=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\ldots$$

<u>Ex:</u> $e=2.71\ldots \underset{z=1}{\approx} 1+1+\frac{1}{2}+\frac{1}{3!}+\frac{1}{4!}+\ldots$

$$z^3e^{2z}=\sum_{n=0}^{\infty} a_n z^n \quad ,\ a_n=\frac{g^{(n)}(0)}{n!}$$

$a_0=0$

$a_1=0$

$a_2=$? product rule !!!

Panel 11

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\Rightarrow e^{(2z)} = 1 + (2z) + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}$$

$$z^3 e^{2z} = z^3 + 2z^4 + \frac{2^2}{2!} z^5 + \frac{2^3}{3!} z^6 + \dots = z^3 \sum \frac{2^n z^n}{n!} = $$

$$= \sum \frac{2^n}{n!} z^{n+3}$$

$$\frac{d^3}{dz^3} \left(z^3 e^{2z}\right)\Big|_{z=0} = 1 \cdot 3! \qquad \frac{d^{15}}{dz^{15}} z^3 e^{2z}\Big|_{z=0} = 2^{12} \cdot 15!$$

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots, \quad a_n = \frac{f^{(n)}(0)}{n!}$$

Panel 12

$$\sin(z) = \cancel{f(0)} + \frac{f'(0)}{1!} z + \cancel{\frac{f''(0)}{2!}} z^2 + \frac{f^{(3)}(0)}{3!} z^3 + \dots$$

$$z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 \pm$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

$\frac{1}{1-z} =$

$e^z =$

$\cos(z) =$

$\sin(z) =$

<u>must know these!</u>