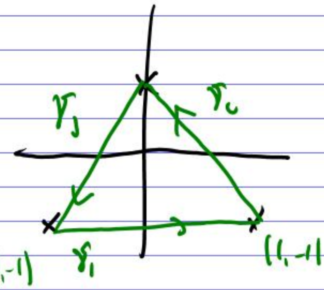
Panel 1

$\int (\bar{z})^2 dz$



γ_1 from $\underline{-1-i}$ to $\underline{1-i}$

$\gamma(t) = -1-i + t(2),\ t \in [0,1]$

$\int_{\gamma_1} (\bar{z})^2 dz = \int_0^1 \overline{(-1-i+2t)}^2\, 2dt = \int_0^1 (-1+i+2t)^2\, 2dt =$ WA

$\int_{\gamma_2} (\bar{z})^2 dz = \int_0^1 \left(1+i+t(-2i-1)\right)^2 (2i-1)dt =$ WA

$\gamma_2(t) = 1-i + t\left(i-(1-i)\right) = 1-i + t(2i-1)$

Panel 2

$\int_a^b f(z)dz$ is generally bad notation.

Because in $\mathbb{C}$ there are inf. many paths from a to b

Unless: It does not matter which path you take.
If f has antideriv.

$\int_a^b f(z)dz = F(b) - F(a)$

$\int_\gamma f(z)dz$ is preferred notation

Panel 3

Review of Big Deal Theorems

1. CR equations
2. Cauchy-Goursat
3. Deform. thm
4. Path indep. (How to)
5. Cauchy Int formula
6. General Cauchy Int. formula

Panel 4

Consequences of (General) Cauchy's Int. Formula

1. f analytic ⇒ f' analytic ⇒ f'' analytic ⇒ $f^{(n)}$ is analytic
   f "once diff'ble always diff'ble"

Gen. Cauchy Int formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}}\, dz$$

$f: \mathbb{R} \to \mathbb{R}$ f' exists ⇏ f'' exists

$f(x) = \sqrt[3]{x} = x^{1/3}$ ⇒ $f' = \frac{1}{3x^{2/3}}$ not diff'ble for $x=0$

$f(x) = x^{4/3}$ ⇒ $f' = \frac{4}{3}x^{1/3}$ but f'' d.n.e at $x=0$

Panel 5

Corollary: $f = u + iv$ analytic then $u, v \in C^\infty$

(7) Morera's Thm: If f is a domain in $\mathbb{C}$ and $\int_C f(z)\,dz = 0$ $\forall$ closed curves in D $\rightarrow$ f analytic

Proof: If $\int_C f\,dz = 0$ $\forall C$ $\rightarrow$ f has antideriv. F. $\rightarrow F' = f$ is again analytic!

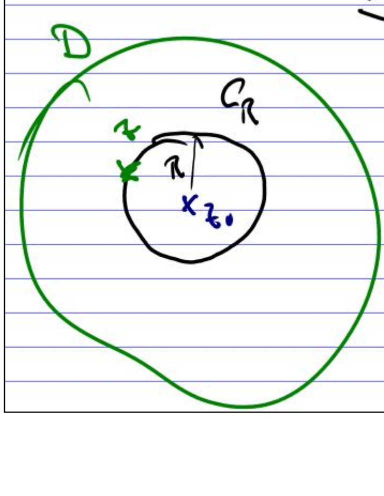
Panel 6

(8) Cauchy's Inequality: If f is analytic inside a circle C_R centered at z_0, and $|f(z)| \le M$ on C_R, then

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}\frac{M}{r^{n+1}} \cdot \text{length}(C_R) = \frac{n!}{\cancel{2\pi}}\frac{M}{r^{n+1}} \cancel{2\pi} r$$

Proof

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$



$$\Rightarrow |f^{(n)}(z_0)| \le \left|\frac{n!}{2\pi i}\int_{C_R}\frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

$$\le \frac{n!}{2\pi}\int_{C_R}\frac{|f(z)|}{|z-z_0|^{n+1}}\,dz \le \frac{n!}{2\pi}\int_{C_R}\frac{M}{r^{n+1}}\,dz \le$$

Panel 7

(8) $|f(z)| \le M$, $\left| f^{(n)}(z_0) \right| \le \dfrac{n!\,M}{R^n}$, $|z-z_0| = R$ & f analytic inside

(9) Liouville's Theorem: If f is entire and bounded, then f is constant

Ex: e^z, $\sin(z)$, $\cos(z)$ are all unbdd!

Proof: Suppose f is entire and bounded by M.
Pick any $z_0 \in \mathbb{C}$, f is analytic on $C_R = |z-z_0| = R$ $\forall R$
Because f is entire!
$\Rightarrow |f'(z_0)| \le \frac{1!}{R} M = \frac{M}{R}$ $(\forall R)$
$f'(z_0) = 0$ $\forall z_0$ $\Rightarrow$ f const.

Panel 8

Fundamental Theorem of Algebra: If $p(z)$ is a polynomial of degree $(n \ge 1)$ then $p(z) = 0$ for at least one $z_0 \in \mathbb{C}$.

(Proof)
$$p_n(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

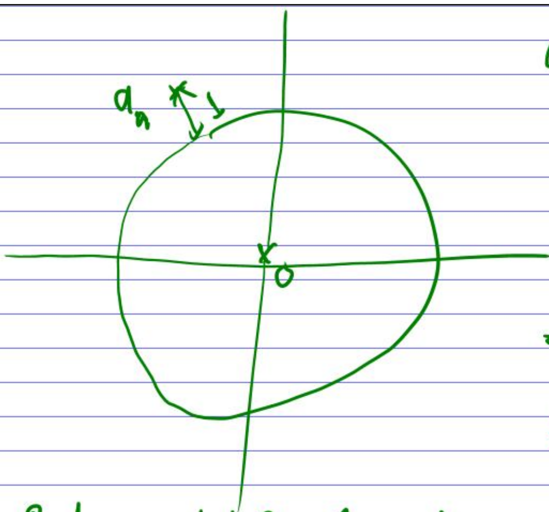
Suppose $p(z) \ne 0$ then $f(z) = \frac{1}{p(z)}$ is entire.

$$f(z) = \frac{1}{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0} = \frac{1}{z^n\left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \dots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right)}$$

Know: $\lim\limits_{z \to \infty} a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} = a_n$

That means if $|z| > R \Rightarrow \left| a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} \right| \le |a_n| - 1$

Panel 9



Know:

$$\lim_{z\to\infty} a_n + \frac{a_{n-1}}{z} + \dots + \frac{a_0}{z^n} = a_n$$

$\Rightarrow$ is R s.t. if $|z|>R$

$\Rightarrow \left|a_n + \dots + \frac{a_0}{z^n}\right| \geq |a_n| - 1$

$\forall |z| > R \Rightarrow |f(z)| < \frac{1}{|a_n|-1}$

But if $|z| \leq R$ then f is cont. on closed, bdd set

$\Rightarrow f$ has a max M $\Rightarrow$ ~~$|f(z)| \leq M \ \forall |z| \leq R$~~

But now $|f(z)| \leq \max\{M, |a_n|-1\}$ $\forall z$! Done!

$f = \text{const}$

Panel 10

Review this proof: $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$

$f(z) = \frac{1}{p(z)} = \frac{1}{z^n\left(a_n + \frac{a_{n-1}}{z} + \dots + a_0/z^n\right)}$ entire if $p \neq 0$

Find R s.t. if $|z| \leq R$, $|f(z)| \leq M$ by cont.

Also, for $|z| > R$ $\Rightarrow |f(z)| = \left|\frac{1}{p(z)}\right| = \frac{1}{|z|^n |a_n + \dots|} \leq \frac{1}{R^n(|a_n|+1)}$

because $\left|a_n + \frac{a_{n-1}}{z} + \dots \frac{a_0}{z^n}\right| \geq |a_n| - 1$

$\Rightarrow f$ entire + bdd $\Rightarrow f = c \Rightarrow \frac{1}{p(z)} = c \Rightarrow p(z) = 1/c$ const.

Panel 11

Consequence of Fund. Thm. of Algebra: If $p_n(z)$ is a complex polynomial of degree n then p has exactly n roots (counting multiplicity)

Proof: $p_n(z) = 0$ has one solution z_1

$\Rightarrow p_n(z) = (z-z_1)\, p_{n-1}(z)$

$p_{n-1}(z) = 0$ has one solution z_2

$p_n(z) = (z-z_1)\, p_{n-1}(z) = (z-z_1)(z-z_2)\, p_{n-2}(z)$

$\vdots$

$p_n(z) = (z-z_1)(z-z_2)\dots(z-z_n)\, c$

Panel 12

There is one more important Theorem, called the Maximum Modulus Principle.

Recall: If f is a function of 2 real variables cont. on a closed bdd set, f has max and min.

Calc

They occur either at a critical point inside set or on boundary. $x^2 - 4xy + y^2$

$x^3 - 2x - 3$, $[1, 2]$

Max Mod Theorem says that in $\mathbb{C}$ it is different $\Rightarrow$ Next time

Panel 13

Foundational Theorems of Complex Analysis

1. Cauchy - Riemann Equations
2. Cauchy - Goursat Thm
3. Deformation thm
4. Path Independence thm
5. Cauchy Integral Formula
6. General Cauchy Int. Formula
7. Morera's thm
8. Cauchy's Inequality
9. Liouville's thm
10. Fund. thm. of Algebra