Panel 1

Integration Theorems

How-To Theorem: f continuous in domain D. Then the following are equivalent:

a) f has antiderivative F

b) $\int_\gamma f(z)\,dz = F(b) - F(a)$

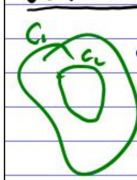
c) $\oint_\gamma f(z)\,dz = 0$ $\forall$ closed curves in D

Cauchy-Goursat Theorem: If f is analytic in a simply connected domain D and f is analytic in D then

$$\int_\gamma f(z)\,dz = 0 \qquad \forall \text{ closed curves } \gamma \text{ in } D$$

Panel 2

Deformation Theorem: If C_1 and C_2 are two



simple closed curves, positively oriented, with C_1 inside C_2, and f analytic between them:

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

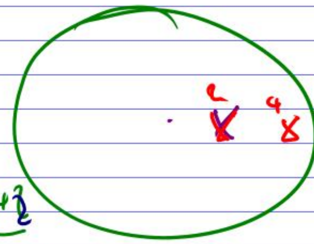
Cauchy's Integration Formula: f analytic inside and on a simple, closed curve C (pos. oriented).

Then $$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i \cdot f(z_0) \qquad \forall z_0 \text{ inside } C$$

$$\left(f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\,dz \right.$$

Panel 3

$$\int_{|z|=5} \frac{z+2}{(z-2)(z-4)}\,dz =$$



$$\frac{A}{z-2} + \frac{B}{z-4} = \frac{A(z-4)+B(z-2)}{(z-2)(z-4)} = \frac{z+2}{(\ )(\ )}$$

$$A(z-4) + B(z-2) = z+2 \qquad \forall z$$

$z=4$: $\quad -2B=6, \quad B=-3$

$z=2$: $\quad 2A=4, \quad A=2$

$$= \int_{|z|=5} \frac{2}{z-2}\,dz + \int_{|z|=5} \frac{-3}{z-4}\,dz = 2\cdot 2\pi i - 3(2\pi i) = \underline{\underline{-2\pi i}}$$

Panel 4

$$\int_\gamma \frac{z+1}{z-0}\,dz \quad f(z), \qquad \gamma \text{ is } |z|=2$$

$$z(t) = 2e^{it}, \quad t\in[0,2\pi]$$

$$dz = 2ie^{it}\,dt$$

① $$\int \frac{2e^{it}+1}{\cancel{2e^{it}}} \cdot \cancel{2}i\cancel{e^{it}}\,dt = i\int_0^{2\pi} (2e^{it}+1)\,dt =$$

$$= i\left(2\frac{1}{i}e^{it}\Big|_0^{2\pi} + t\Big|_0^{2\pi}\right) =$$

$$= i\left(2(e^{2\pi i} - e^0) + 2\pi\right) = \underline{\underline{2\pi i}}$$

② $$\int_{|z|=2} \frac{z+1}{z}\,dz = (0+1)\cdot 2\pi i = \underline{2\pi i}$$

Panel 5

$\int (\bar{z})^2 dz$

No shortcut: ∃ integral, parametrize lines, ...

$\int_{-1}^{2} z^2+3\,dz$

No curve specified!!!
Curve cannot matter
True if f analytic

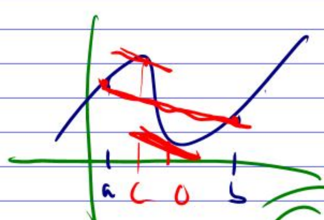
$\int_{-1}^{2} \bar{z}^2+3\,dz$

No Sense!

Panel 6

Mean Value Thm:

$\frac{f(b)-f(a)}{b-a} = f'(c), \quad c \in (a,b)$



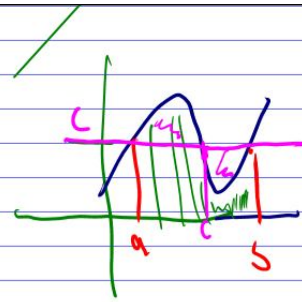
New $z(t) = e^{it}, \quad t \in [-\pi, \pi]$

$0 =$ $\frac{z(\pi)-z(-\pi)}{\pi+\pi} = f'(c) = ie^{ic}$

$z(\pi) = e^{i\pi} = -1, \quad z(-\pi) = e^{-i\pi} = -1$

Panel 7

MVT for integration.



$$\int_a^b f(x)\,dx = c(b-a)$$

$$= f(c)(b-a)$$

$$\int_{-\pi}^{\pi} e^{it}\,dt = \frac{1}{i}e^{it}\Big|_{-\pi}^{\pi} = 0$$

$f(c)(\pi+\pi)$

$= e^{ic}\,2\pi \neq 0$ again!

Panel 8

Next time: prove Cauchy Int. Formula

Iclw

Panel 9

Cauchy's Int. Formula rephrased:

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \quad \forall z_0 \in D$$

Replace z_0 by ω: $f(\omega) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-\omega}\, d(z) \quad \forall \omega \in D$

This is a function.

$$\frac{\partial f}{\partial w} = \frac{\partial}{\partial w}\left(\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\, dz\right) = \frac{1}{2\pi i}\int_\gamma \frac{\partial}{\partial w} \frac{f(z)}{z-w}\, dz =$$

$$= \frac{1}{2\pi i}\int_\gamma - f(z)(z-w)^{-2}(-1)\, dz$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^2}\, dz$$

Panel 10

$$f(w) = \frac{1}{2\pi i}\int \frac{f(z)}{z-w}\, dz$$

$$f'(w) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-w)^2}\, dz$$

$$\Rightarrow f''(w) = \frac{1}{2\pi i}\int \frac{f(z)}{(z-w)^3}\cdot 2\, dz = \frac{2}{2\pi i}\int \frac{f(z)}{(z-w)^3}\, dz$$

$$f'''(w) = \frac{3\cdot 2}{2\pi i}\cdot\int \frac{f(w)}{(z-w)^4}\, dz = \frac{3!}{2\pi i}\int \frac{f(w)}{(z-w)^4}\, dz$$

$$f''''(w) = \frac{4!}{2\pi i}\int \frac{f(w)}{(z-w)^5}\, dz$$

$$\rightarrow f^{(n)}(w) = \frac{n!}{2\pi i}\int \frac{f(z)}{(z-w)^{n+1}}\, dz$$

Panel 11

General Cauchy Integral Formula: f analytic inside and on a simple, closed curve C (pos. oriented). Then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

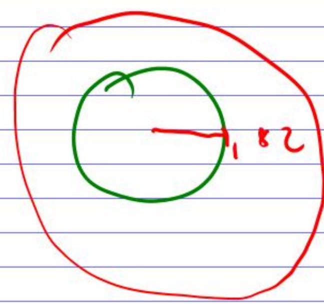
Ex: $\displaystyle\int_{|z|=1} \frac{e^{2z}}{z^4}\,dz = \frac{2\pi i}{n!} f^{(3)}(0) = \frac{2\pi i}{3!}\, 2^3 e^{2z}\Big|_{z=0} = \frac{2\pi i}{3!}\, 8$

$$\int_{|z|=1} e^{2z}\,dz = 0$$

$$\int_{|z|=1} \frac{e^{2z}}{z-0}\,dz = 2\pi i\, f(0) = 2\pi i\, e^{2\cdot 0} = 2\pi i$$

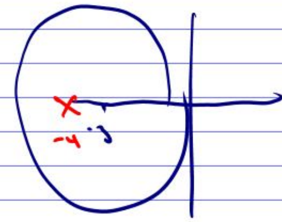
Panel 12

$$\int_{|z|=R} \frac{\sin(z)}{(z-2)(z+4)^2}\,dz = 0$$



$R=1$: 0

$R=3$: $2\pi i \dfrac{\sin(2)}{6}$



$|z+3|=2$

$$\frac{2\pi i}{1!}\frac{d}{dz}\frac{\sin(z)}{z-2}\Big|_{z=-4}$$